CORRECTIONS

Page	Line	
23	3	$I = \{1,2,\ldots,n\}$
26	17	or, (ii) $A\phi$ is a regular product of the B_μ, $\mu \in I\Gamma$, and there exists,
27	19	taken from [18], Theorem 3.4.
33	6	It can be proved by induction on m that,
	7	delete - ([6] Chapter II, lemma 6.5)
35	3,4	in $\overline{A}_\lambda$, $\overline{A}_\mu$
37	11	$C_{m+1}(F_i)\alpha_m$
40	17	$R_2 = \mathrm{sgp}\,(y_1{}^3, y_2{}^3, \ldots$
42	17	$[v_2,v_1,v_r]$
46	12	$[x_i, x_j, (p-1)x_i][x_i, px_j]^{-1}$
	14	$p^{n_i}!/(p^{n_i} - p)!p!$
47	13	$[x_i, x_j, (p-1)x_i]$
	16	β_1
48	14	$b' =(ab^{-1})^q$
52	18	$\mathrm{sgp}(X_j \mid j \neq i)$
	24	$j \neq i$
54	2	$x = b_1{}^{m_1}b_2{}^{m_2} \ldots$
56	13	$b_r = 1$
	15	$[b_s, b_t]$
86	18	$b_3,b_4 \in B$; $b_3 \neq b_4$;
99	4	Soc.(2) <u>36</u>
	7	Hopf
	14	OGIZ
100	4	endlichen
	5	lineare
	8	ditto.

THE MULTIPLICATOR OF VARIOUS PRODUCTS OF GROUPS

by

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A thesis presented for the degree of Doctor of Philosophy
at the Australian National University

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Dedicated

To the Memory of

HANNA NEUMANN

The results presented in this thesis are my own except where otherwise stated.

William Haebich

William Haebich

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CONTENTS

CHAPTER 1

PRELIMINARIES

1. Introduction

Most known results relating a group to its multiplicator are "non structural" in the sense that they relate various numerical invariants of group and multiplicator. This means that the calculation of a specific multiplicator often involves lengthy, specialised arguments. Standard product constructions like the direct and wreath products provide a way of describing certain group structures. When G is a given product of the groups A_λ, $\lambda \in I$, its multiplicator M(G) is necessarily dependent on the A_λ. We are led to ask whether M(G) is also a recognisable product of the A_λ, or perhaps the $M(A_\lambda)$. Until recently the only known relationship of this sort was a classical result due to Schur [23]: if G is the direct product, $A_1 \times A_2$, then M(G) is isomorphic to $M(A_1) \times M(A_2) \times (A_1 \otimes A_2)$. In this thesis an attempt is made to furnish some useful "structural" theorems on the multiplicator by establishing similar results for various other products.

All products dealt with are the type where G is generated by the A_λ, or (multiple) isomorphic copies of each A_λ, $\lambda \in I$. M(G) can be characterised as $R \cap F'/[R,F]$ where F/R is a presentation for G (see section 2). From this point of view, the most direct way of approaching M(G) is to try to express $R \cap F'/[R,F]$ in terms of the $R \cap F'/[(R \cap F_\lambda), F_\lambda]$ where F_λ is the group of elements in F which map onto A_λ. ($F_\lambda / R \cap F_\lambda$ is a presentation for A_λ.) In fact this is the method applied by Schur to the direct product. The difficulty in extending it to more complicated products lies in not knowing how the F_λ will generate F. To get around this problem we use a technique that is essentially the reverse of the direct approach. We start with presentations F_λ/R_λ for each A_λ and use them to <u>construct</u> an appropriate presentation for G. With one exception, it is best just to take the free product $F = \prod^*_{\lambda\in I} F_\lambda$ and to find the kernel of the natural epimorphism from F onto G.

The most complete results are obtained for regular products and verbal wreath products. In Chapter 2 it is shown that, if G is a regular product of the A_λ, $\lambda \in I$, then M(G) is the direct product of the $M(A_\lambda)$ together with another group.

Chapter 3 is concerned with the consequences of this result for the particular case where G is a nilpotent product. We formulate some "reduction" theorems and calculate the multiplicator explicitly for certain finite nilpotent products of cycles. In doing so we offer a somewhat simplified proof of a result due to Struik [26], [27].

Splitting extensions (semi direct products) have less restrictions on their factors than regular products and produce a correspondingly weaker result. We prove in Chapter 4 that if G is a splitting extension of A by B then M(B) is a direct summand of M(G). Moreover, when G is a verbal wreath product, $Awr_V B$, it is shown that M(G) is isomorphic to the direct product of M(B), M(A) and another group.

The main theorem of Chapter 4 can be pushed further in another direction. Chapter 5 outlines a proof to the effect that the complement of M(B) in M(G) is a certain central product when A is a finitely generated abelian group with no elements of even order. Also the multiplicator of a splitting extension of one finite cycle by another is calculated explicitly.

Finally, in Chapter 6, we indicate the limit of effectiveness of our stated technique by applying it to a central product. The sole difference between a central product of a pair of groups and their direct product is that the factors need not intersect trivially and yet the usual analysis of the multiplicator seems to break down.

A big advantage of the technique is that most of the results obtained by it do not require any conditions of finiteness. When the products involved are finite, it enables us to calculate representing groups for them or at least indicate properties of the representing groups. Representing groups play a significant part in multiplicator theory and are even more difficult to calculate than the multiplicators themselves.

During the writing of this thesis three papers have come to light which independently establish some of its results. Tahara [28] has proved the main theorem on the multiplicator of a splitting extension. He also obtains the result on the splitting extension of a pair of cycles. Blackburn [2] finds the multiplicator of an ordinary wreath product using a more complicated argument, and Evens [4] establishes a stronger version of the chief result of Chapter 5, under the severe restriction that G be a p-group. All three use homological methods. Our purely group theoretical attack gives a unified approach to the various products, provides some shorter proofs and more general results and perhaps gives more insight to the problem by closely linking the group structure with the multiplicator of the group.

2. Some characteristics and applications of the multiplicator

The multiplicator first arose as a part of the theory of group representations. It is associated with the problem of finding a relationship between the representations of a finite group G, over a field $\underline{F}$, and the representations of a given central extension of G.

Actually it is the so called <u>projective representations</u> which are most relevant to this question. They are defined in terms of factor sets of G with coefficients in the multiplicative group of $\underline{F}$: that is, mappings α from the pairs $G \times G$ to $\underline{F}$ which satisfy the following identity,

$$(g_2,g_3)\alpha\ ((g_1g_2,g_3)\alpha)^{-1}\ (g_1,g_2g_3)\alpha\ ((g_1,g_2)\alpha)^{-1} = 1$$

for all $g_1, g_2, g_3 \in G$. This fits in with the fact that the possible central extensions of an abelian group, M, by G are determined by the factor sets of G with coefficients in M.

Suppose (L,M) is a pair of groups such that

(i) L is finite.

(ii) $M \leq L' \cap Z(L)$.

(iii) $L/M \simeq G$.

We will adopt the terminology of Jones and Wiegold [14] and call these <u>defining pairs</u> for G. A one to one correspondence can be constructed between the (equivalence classes of) irreducible representations of L and the irreducible projective representations of G [14] Kapitel V, §24. Schur [22] proved that, over all defining pairs (L,M), there exists an M of maximal order and that it is unique up to isomorphism. He called this finite abelian group the multiplicator of G.

We will refer to a defining pair (L,M), with M maximal, as a <u>representing pair</u> and to L as a <u>representing group</u> for G. Representing groups for a given group are generally not unique.

The multiplication in $\underline{F}$ can be used to impose a group structure on the set of factor sets defined above. If $\underline{F}$ is algebraically closed of characteristic zero, then Schur's multiplicator is isomorphic to a certain quotient group of this group, which turns out to be the second cohomology group $H^2(G,\underline{F})$ (where G has trivial action on $\underline{F}$), [14] Kapitel V, §23. $H^2(G,\underline{F})$ is an abelian group than can be constructed for both finite and infinite G and so it provides a natural extension of Schur's original definition of the multiplicator to encompass all groups.

The second homology group $H_2(G,Z)$ is often used as an alternative to $H^2(G,\underline{F})$. Denote $\mathrm{Hom}(A,\underline{F})$ by A^*. Then these are dual groups in the sense that $H^2(G,\underline{F})$ is isomorphic to $H_2(G,Z)^*$ ([8] Chapter 3, Theorem 3) and $(A^*)^*$ is isomorphic to A^* for all abelian groups, A. $H_2(G,Z)$ is a legitimate extension of Schur's definition because it is actually isomorphic to $H^2(G,\underline{F})$ when G is finite ($A^* \simeq A$ for A finite abelian).

Hopf [13] showed that $H_2(G,Z)$ is isomorphic to $R \cap F'/[R,F]$ where F/R is a presentation for G. <u>It is this characterisation of M(G) which will be used throughout the thesis</u>.

We have indicated that the projective representations of a finite group G determine the ordinary representations of certain central extensions by G. Clearly if M(G) is trivial then all such extensions are trivial and the projective and ordinary representations of G coincide. In general, knowledge of the structure of M(G) facilitates calculation of the projective representations of G. This feature of the multiplicator has been of some use in the current investigations of finite simple groups [3], [25].

Applications outside representation theory come from the connection of the structure of the multiplicator with several group invariants. A typical example is the deficiency, d(G), of a finite group G. Given a presentation for G on n generators with r defining relations, d(G) is defined to be the minimum possible value of r-n, and this is bounded below the minimum number of elements needed to generate M(G) [14] Kapitel V, §25.

For a finite group G, Green [7] established a relationship between AutG and the multiplicator of any given subgroup of G. He calculated a bound on the order of the multiplicator of a p-group and was then able to put a bound on the order of AutG for G a p-group. This led Wiegold [30] to find a bound on $|G'|$ in terms of $|G/Z(G)|$ where the latter is a prime power. Improvements have been made by Gäschutz, Neubüser and Ti Yen [5] and again by Wiegold [31]. A more detailed survey of this work is contained in the introduction of [31].

These examples are by no means exhaustive.

3. Notation

Most proofs involve common group theoretical arguments and very little special notation is necessary. Capital Roman letters are used for sets, groups and matrices, and small Roman letters for their elements and entries. Mappings are denoted by small Greek letters. The end of a proof will be marked by a pair of lines ... //. We list below the symbols and conventions most widely used. Other isolated pieces of notation will be defined in the text as they are needed.

Z	the ring of integers
i, j, k, ℓ, m, n (and sometimes r, s, t)	non negative integers
p	a prime
E	the trivial group
1	the identity of any group
$F; F_1, F_2, \ldots; F_\lambda$	free groups
$\lvert G\rvert$	the order of the group G when G is finite
G'	the derived group of G
$M(G)$	the multiplicator of G

$Z(G)$	the centre of G
$\mathrm{Aut}G$	the group of automorphisms of G
$\Phi(G)$	the Frattini subgroup of G
$V(G)$	a verbal subgroup of G. (Only V will be used for this. The symbol $R(G)$, for example, does not stand for a verbal subgroup)
$\gamma_n(G)$	the n^{th} term of the lower central series of G
A^G	the normal subgroup of G generated by a subset (subgroup), A
$A \leq B$	A is a subgroup of B
$A \trianglelefteq B$	A is a normal subgroup of B
$sgp(a, b, c, \ldots)$	the subgroup generated by the elements $a, b, c, \ldots$
$sgp(A, B, C, \ldots)$	the subgroup generated by the subsets (subgroups) $A, B, C, \ldots$
a^b	the product of elements $b^{-1}ab$ in a group
a^{-b}	the inverse of a^b
$[a,b]$	the commutator $a^{-1}b^{-1}ab$
$[a_1, a_2, \ldots, a_n]$	the left normed commutator $[[\ldots[[a_1,a_2],a_3]\ldots],a_n]$
$[a,nb]$	$[a,\underbrace{b,b,\ldots,b}_{n}]$
$[A,B]$	the commutator subgroup, $sgp([a,b] \mid a \in A,\ b \in B)$
$A * B$	the free product of A and B
$A \otimes B$	the tensor product of A and B
$\mathrm{Hom}(A,B)$	the group of homomorphisms from A to B
$SG(A,B)$	the complement of $M(B)$ in $M(G)$ where G is a splitting extension of A by B
I	an ordered set used as an index set
λ,μ (and occasionally ρ,σ)	elements of I
$\prod^*_{\lambda\in I} A_\lambda$	the free product of the A_λ, $\lambda \in I$

$\Pi^{\times}_{\lambda\in I} A$ the direct product of the A_λ, $\lambda \in I$

For the A_λ, $\lambda \in I$, subgroups of G ...

$[A_\lambda]$ the cartesian, $sgp([a_\lambda, a_\mu] \mid \lambda, \mu \in I; \lambda \neq \mu)$ in G

$\Pi_{\lambda\in I} A_\lambda$ the set of products $a_{\lambda_1} a_{\lambda_2} \ldots a_{\lambda_n}$ in G where $\lambda_1 < \lambda_2 < \ldots < \lambda_n$

$C_m(A_\lambda)$ $\gamma_m(G) \cap [A_\lambda{}^G]$ where $G = \Pi^*_{\lambda\in I} A_\lambda$

$MG(A_\lambda)$ the complement of $\Pi^{\times}_{\lambda\in I} M(A_\lambda)$ in M(G) where G is a regular product of the A_λ.

4. <u>Basic results</u>

Propositions (1.3.1) to (1.3.11) are standard commutator identities taken from lists in Magnus, Karrass and Solitar [17], pages 290 and 293; Hall [10], page 150 and Golovin [6], 2.1. The results on commutator subgroups are from [6] and the final three propositions are bits of elementary group theory used frequently in most chapters - often without explicit reference. Another selection of results, also quoted from [6], fits more properly into Chapter 2 after a preliminary discussion of regular products.

(1.3.1) $a^b = a[a,b]$

(1.3.2) $[a,b]^{-1} = [b,a]$

(1.3.3) $[a,b^{-1}] = [b,a]^{b^{-1}}$

(1.3.4) $[ab,c] = [a,c]^b[b,c]$

$= [a,c][a,c,b][b,c]$

(1.3.5) $[a,bc] = [a,c][a,b]^c$

$= [a,c][a,b][a,b,c]$

(1.3.6) $[a,b^{-1},c]^b[b,c^{-1},a]^c[c,a^{-1},b]^a = 1$

If $a \in \gamma_\ell(G)$, $b \in \gamma_m(G)$ and $c \in \gamma_n(G)$ (the lower central series is defined in Chapter 3, section 1) then we have,

(1.3.7) $ab \equiv ba \mod \gamma_{\ell+m}(G)$

(1.3.8) $[ab,c] \equiv [a,c][b,c] \mod \gamma_{\ell+m+n}(G)$

(1.3.9) $[a,bc] \equiv [a,b][a,c] \mod \gamma_{\ell+m+n}(G)$

(1.3.10) $[a^{-1},b] = [a,b]^{-1} \mod \gamma_{2\ell+m}(G)$

(1.3.11) $[a,b,c][b,c,a][c,a,b] \equiv 1 \mod \gamma_{\ell+m+n+1}(G)$

Suppose that A, B and C are subgroups of G, then ...

(1.3.12) $A^B = A[A,B]$

(1.3.13) $[A,B] = [B,A]$

(1.3.14) If $A \leq C$ then $[A,B] \leq [C,B]$

(1.3.15) $[A^G,G] = [A,G]^G = [A,G]$

(1.3.16) Suppose A and B are generated by sets X and Y. Then [A,B] is generated by the elements $[x,y]^c$ where $x \in X$, $y \in Y$ and $c \in sgp(A,B)$. If B is normal in G then $[A,B]$ is generated by the elements $[x,b]$ where $x \in X$, $b \in B$.

(1.3.17) If B is normal in G and each N_λ is normal in G, $\lambda \in I$, then

$$[(\prod_{\lambda\in I} N_\lambda)A,B] = (\prod_{\lambda\in I} [N_\lambda,B])[A,B]$$

(1.3.18) If $B \leq C$ and either A or B is normal in G then $(AB\cap C) = (A\cap C)B$.

(1.3.19) If ϕ is a homomorphism of G whose kernel lies in A then, $(A\cap B)\phi = A\phi \cap B\phi$.

(1.3.20) Suppose B and $\bar{B}$ are normal subgroups of A. If ϕ and $\bar{\phi}$ are homomorphisms of G such that $(A \cap \ker\phi)B = (A \cap \ker\bar{\phi})\bar{B}$ then, the map $(a\phi)B\phi \mapsto (a\bar{\phi})(\bar{B}\bar{\phi})$, $a \in A$, is an isomorphism of $A\phi/B\phi$ and $A\bar{\phi}/\bar{B}\bar{\phi}$.

Proof Now,

$$\begin{aligned} A\phi/B\phi &\simeq \frac{A\ker\phi/\ker\phi}{B\ker\phi/\ker\phi} \\ &\simeq A\ker\phi/B\ker\phi \\ &= AB\ker\phi/B\ker\phi \\ &\simeq A/A \cap B\ker\phi \\ &= A/(A \cap \ker\phi)B \end{aligned}$$

and similarly $A\bar{\phi}/\bar{B}\bar{\phi} \simeq A/(A \cap \ker\bar{\phi})\bar{B}$. A careful check of these isomorphisms shows that individual elements are mapped in the required manner. //

CHAPTER 2

THE MULTIPLICATOR OF A REGULAR PRODUCT

1. Regular products

We begin by making a distinction between the use of the words "product" and "multiplication" that is tacitly employed by Moran [18]. In general, any group generated by a set of groups (or isomorphic copies of them) is called a product of those groups. A multiplication, on the other hand, is a rule which assigns a unique group to every arbitrary, ordered set of groups such that the assigned group is a product of the groups in the set. The groups from which a given product is formed are called the factors of the product.

The free multiplication and direct multiplication of a set of groups have, amongst others, the following three properties in common.

(i) The multiplication is commutative in the sense that it assigns the same product to a given set of groups no matter how the set is ordered.

(ii) The multiplication is associative in the sense that, if an arbitrary set of groups is partitioned into disjoint subsets, the product, under the multiplication, of the groups assigned to each subset by the multiplication is equal to the product assigned to the whole set.

(iii) Any factor in the free product and the direct product of a set of groups intersects the normal closure of the other factors in the product trivially.

The question as to which other multiplications satisfy these properties was originally put by Kuroš [16] (page 350) and subsequently investigated by Golovin [6] and Moran [18], [19]. As a prelude to constructing a new class of multiplications, Golovin defined regular products in terms of property (iii).

DEFINITION 2.1.1 G is a regular product of its subgroups A_λ, $\lambda \in I$, where I is an ordered set, if they generate G and $A_\lambda \cap \hat{A}_\lambda = E$ where $\hat{A}_\lambda = \mathrm{sgp}(A_\mu{}^G | \mu \in I, \mu \neq \lambda)$. The A_λ are called regular factors of G. Any multiplication which takes every ordered set of groups to a regular product of those groups is called a regular multiplication.

The general question of associativity remains unsettled although Moran has constructed wide classes of associative and non associative regular multiplications. We turn our attention to the regular products themselves and list some properties due to Golovin that will be needed in section 2. Suppose that G is generated by its subgroups A_λ.

(2.1.2) $[A_\lambda{}^G]$ is normal in G and $[A_\lambda{}^G] = [A_\lambda]^G$.

(2.1.3) In general $G = (\prod_{\lambda \in I} A_\lambda)[A_\lambda{}^G]$ and G is a regular product if and only if each of its elements can be written uniquely as a product $a_{\lambda_1} a_{\lambda_2} \ldots a_{\lambda_n} u$ where $a_{\lambda_i} \in A_{\lambda_i}$, $\lambda_1 < \lambda_2 < \ldots \lambda_n$ and $u \in [A_\lambda{}^G]$.

(2.1.4) If G is a regular product and X_λ is a subgroup of A_λ for each $\lambda \in I$, then the subgroup of G generated by the X_λ is a regular product of the X_λ.

(2.1.5) If G is a regular product then $G' = (\prod_{\lambda \in I} A_\lambda{}')[A_\lambda{}^G]$.

DEFINITION 2.1.6 If G is a regular product, a homomorphism ϕ of G to a group $\bar{G}$ is a regular homomorphism if $\ker\phi$ is a subgroup of $[A_\lambda{}^G]$: hence the terms regular homomorphic image and regular quotient group.

(2.1.7) If G is a regular product and $\phi : G \to \bar{G}$ is a regular homomorphism then ϕ restricted to A_λ is an isomorphism for each $\lambda \in I$ and $G\phi$ is a regular product of the $A_\lambda\phi$.

(2.1.8) If $\psi : \prod^*_{\lambda\in I} A_\lambda \to G$ is the natural homomorphism induced by the identity map on each A_λ, then G is a regular product if and only if ψ is regular.

This last result shows that the free product of a set of groups can be interpreted as the "largest" regular product of those groups. In the same way, their direct product is the "smallest" such product. For, if G is a regular product of the A_λ, then the quotient $G/[A_\lambda{}^G]$ is a direct product of copies of the A_λ.

2. Calculation of the multiplicator

Suppose that G is a regular product of the A_λ. The plan is to calculate M(G) by finding a presentation for G in terms of presentations for each A_λ. In fact we begin with greater generality and let B_λ be a fixed group which maps epimorphically onto A_λ under ν_λ. This will facilitate the construction of a representing group for G when G is finite.

Suppose that C_λ is the kernel of ν_λ and that ν is the natural epimorphism from the free product $B = \prod^*_{\lambda\in I} B_\lambda$ onto $A = \prod^*_{\lambda\in I} A_\lambda$ induced by the ν_λ. Further, if ψ is the natural homomorphism from A onto G induced by the identity on each A_λ, let H be the kernel of ψ and H_0 the group in B which maps onto H under ν. We have that

$$\begin{aligned}
H &= H \cap [A_\lambda{}^A] && \text{since } H \le [A_\lambda{}^A] \text{ by (2.1.8)} \\
&= H_0\nu \cap [B_\lambda{}^B]\nu && \text{since } [B_\lambda{}^B]\nu = [B_\lambda\nu^{B\nu}] \text{ by} \\
& && \text{definition of the cartesian} \\
&= (H_0 \cap [B_\lambda{}^B])\nu && \text{since } \ker\nu \le H_0 \text{ by definition} \\
&= K\nu && \text{where } K = H_0 \cap [B_\lambda{}^B] \text{ is normal in B.}
\end{aligned}$$

LEMMA 2.2.1 B/C is isomorphic to G where $C = (\prod_{\lambda\in I} C_\lambda{}^B)K$.

Proof (i) We first show that $\ker\nu = \prod_{\lambda\in I} C_\lambda{}^B$. (This is almost certainly well known.) The kernel of ν contains $\prod_{\lambda\in I} C_\lambda{}^B$ since, by construction of ν, $C_\lambda\nu$ is trivial for each $\lambda \in I$. On the other hand,

$$\begin{aligned}
B_\mu \cap \prod_{\lambda\in I} C_\lambda{}^B &= B_\mu \cap D_\mu C_\mu{}^B && \text{where } D_\mu = \prod_{\substack{\lambda\in I\\ \lambda\neq\mu}} C_\lambda{}^B \\
&= (B_\mu \cap D_\mu[C_\mu, B])C_\mu && \text{since } C_\mu \le B_\mu \\
&= C_\mu && \text{since} \\
& && B_\mu \cap D_\mu[C_\mu, B] \le B_\mu \cap \hat{B}_\mu .
\end{aligned}$$

Thus, if ρ is the canonical homomorphism from B onto $B/\prod_{\lambda\in I} C_\lambda{}^B$, then $B_\lambda\rho$ is isomorphic to B_λ/C_λ, and hence to A_λ.

Now $B\rho$ is generated by the $B_\lambda\rho$, $\lambda \in I$. Let η be the natural epimorphism from A onto $B\rho$ induced by the isomorphism from each A_λ to $B_\lambda\rho$. The following diagram commutes

$$\begin{array}{ccc}
B & \xrightarrow{\rho} & B/\prod_{\lambda\in I} C_\lambda{}^B \\
\nu\downarrow & \nearrow_{\eta} & \\
A & &
\end{array}$$

The kernel of ν is therefore contained in $\ker\rho = \prod_{\lambda\in I} C_\lambda^{\ B}$ and hence $\ker\nu = \prod_{\lambda\in I} C_\lambda^{\ B}$.

(ii) The product, $\nu\psi$, of mappings is an epimorphism from B onto G. Now,

$$
\begin{aligned}
b \in \ker\nu\psi &\iff b\nu\psi = 1 \quad \text{in } G \\
&\iff b\nu \in H \quad \text{since } H = \ker\psi \\
&\iff b \in \ker\nu K \quad \text{since } K\nu = H .
\end{aligned}
$$

But
$$
\begin{aligned}
\ker\nu K &= \Big(\prod_{\lambda\in I} C_\lambda^{\ B}\Big)K \quad \text{from (i)} \\
&= C .
\end{aligned}
$$

Thus
$$C = \ker\nu\psi \quad \text{from (i) and (ii).} \qquad //$$

We aim to find the structure of $C\cap B'/[C,B]$ which is isomorphic to $M(G)$ when B is free by Lemma 2.2.1.

LEMMA 2.2.2 $\prod_{\lambda\in I} C_\lambda^{\ B} = \Big(\prod_{\lambda\in I} C_\lambda\Big)D$ in B where $D = \prod_{\substack{\lambda,\mu\in I\\ \lambda\neq\mu}} [C_\lambda, B_\mu]^B$.

Proof We prove that $c_\lambda^{\ b}$ is in $C_\lambda D$ for all $c_\lambda \in C$ and $b \in B$ by induction on the length of the normal form for b. This is trivial when b is of length one, for $c_\lambda^{\ b_\mu} = c_\lambda[c_\lambda, b_\mu]$, $\lambda \neq \mu$, and $c_\lambda^{\ b_\lambda} \in C_\lambda$ since C_λ is normal in B. Suppose that $c_\lambda^{\ b} \in C_\lambda D$ for all elements b of length m and that bb_μ is a reduced word in B. Then,

$$
\begin{aligned}
c_\lambda^{\ bb_\mu} &= (c_\lambda' d)^{b_\mu} \quad \text{where } c_\lambda' \in C,\ d \in D \text{ by the inductive hypothesis} \\
&= c_\lambda'[c_\lambda', b_\mu] d^{b_\mu} \in C_\lambda D \quad \text{since } [c_\lambda', b_\mu] \in D \text{ for } \lambda \neq \mu, \text{ and if} \\
&\qquad \lambda = \mu, \text{ then } [c_\lambda', b_\lambda] \in C_\lambda .
\end{aligned}
$$

Thus $C_\lambda^{\ B}$ is a subgroup of $C_\lambda D$. It follows that $\prod_{\lambda\in I} C_\lambda^{\ B}$ is a subgroup of $\prod_{\lambda\in I}(C_\lambda D)$. By definition, every element of $\prod_{\lambda\in I}(C_\lambda D)$ can be written in the form $c_{\lambda_1}d_{\lambda_1}c_{\lambda_2}d_{\lambda_2}\ldots c_{\lambda_n}d_{\lambda_n}$ where $c_{\lambda_i}\in C_{\lambda_i}$, $d_{\lambda_i}\in D_{\lambda_i}$ and $\lambda_1<\lambda_2<\ldots<\lambda_n$. But,

$$c_{\lambda_1}d_{\lambda_1}c_{\lambda_2}d_{\lambda_2}\ldots c_{\lambda_n}d_{\lambda_n}$$

$$= (c_{\lambda_1}c_{\lambda_2}\ldots c_{\lambda_n})d_{\lambda_1}^{\ (c_{\lambda_2}c_{\lambda_3}\ldots c_{\lambda_n})}d_{\lambda_2}^{\ (c_{\lambda_3}\ldots c_{\lambda_n})}\ldots d_{\lambda_n}$$

which belongs to $(\prod_{\lambda\in I} C_\lambda)D$. Hence $\prod_{\lambda\in I} C_\lambda^{\ B}$ is a subgroup of $(\prod_{\lambda\in I} C_\lambda)D$. The reverse inclusion is trivial since $[C_\lambda,B_\mu]^B$ is a subgroup of $C_\lambda^{\ B}$.//

(2.2.3) (i) $C\cap B' = (\prod_{\lambda\in I}(C_\lambda\cap B_\lambda'))DK$.

(ii) $[C,B] = (\prod_{\lambda\in I}[C_\lambda,B_\lambda])D[K,B]$.

Proof (i) C consists of all elements of the form $c = c_{\lambda_1}c_{\lambda_2}\ldots c_{\lambda_m}dk$ where $c_{\lambda_i}\in C_{\lambda_i}$, $d\in D$, $k\in K$, by the preceding two lemmas. The fact that DK is in B' means that c belongs to B' if and only if $\bar{c} = c_{\lambda_1}c_{\lambda_2}\ldots c_{\lambda_m}$ belongs to B'. Now, by (2.1.5), every element of B' is of the form $b_{\lambda_1}'b_{\lambda_2}'\ldots b_{\lambda_n}'u$ where $b_\lambda'\in B_\lambda'$, $u\in[B_\lambda^{\ B}]$. But both C_{λ_i} and B_{λ_i}' are subgroups of B_{λ_i}. Therefore $\bar{c}$ is in B' if and only if c_{λ_i} is in B_{λ_i}' by (2.1.2).

(ii) $$\begin{aligned}[C,B] &= [(\prod_{\lambda\in I} C_\lambda^{\ B})K,B]\\ &= (\prod_{\lambda\in I}[C_\lambda^{\ B},B])[K,B] \quad \text{by (1.3.17)}\\ &= (\prod_{\lambda\in I}[C_\lambda,B]^B)[K,B] \quad \text{by (1.3.15)} \quad - \quad (*)\ .\end{aligned}$$

Now $[C_\lambda, B]$ is generated by the elements $[c_\lambda, b]$ where $c_\lambda \in C_\lambda$, $b \in B$. We can prove that $[c_\lambda, b] \in [C_\lambda, B_\lambda]D$ by induction on the length of b precisely as in Lemma 2.2.2. The result is trivial for $b = b_\mu$. Suppose that $[c_\lambda, b] \in [C_\lambda, B_\lambda]D$ for b of length m, then $[c_\lambda, b] = x_\lambda d$ where $x_\lambda \in [C_\lambda, B_\lambda]$ and $d \in D$. If bb_μ is reduced then,

$$\begin{aligned}
[c_\lambda, bb_\mu] &= [c_\lambda, b_\mu][c_\lambda, b]^{b_\mu} \\
&= [c_\lambda, b_\mu]x_\lambda^{b_\mu} d^{b_\mu} \\
&= [c_\lambda, b_\mu]x_\lambda[x_\lambda, b_\mu]d^{b_\mu} \quad \text{which is an element of} \\
&\qquad\qquad [C_\lambda, B_\lambda][C_\lambda, B_\mu]D.
\end{aligned}$$

Thus $[C_\lambda, B]$ is a subgroup of $[C_\lambda, B_\lambda]D$. Consequently,

$$\begin{aligned}
[C_\lambda, B]^B &\le [C_\lambda, B_\lambda]^B D^B \\
&= [C_\lambda, B_\lambda][[C_\lambda, B_\lambda], B]D \quad \text{by (1.3.12) and since } D \trianglelefteq B \\
&\le [C_\lambda, B_\lambda][C_\lambda, B]D \qquad \text{since } C_\lambda \trianglelefteq B_\lambda \\
&\le [C_\lambda, B_\lambda]D\,.
\end{aligned}$$

Hence, from (*), $[C,B]$ is a subgroup of $(\prod_{\lambda\in I}([C_\lambda, B_\lambda]D))[K,B]$.

Again, copying the argument of Lemma 2.2.2, every element of $\prod_{\lambda\in I}([C_\lambda, B_\lambda]D)$ is a product of the form $x_{\lambda_1}d_{\lambda_1}x_{\lambda_2}d_{\lambda_2}\ldots x_{\lambda_n}d_{\lambda_n}$ where $x_{\lambda_i} \in [C_{\lambda_i}, B_{\lambda_i}]$, $d_{\lambda_i} \in D$ and $\lambda_1 < \lambda_2 < \ldots < \lambda_n$. This product can be rearranged to show that it lies in $(\prod_{\lambda\in I}[C_\lambda, B_\lambda])D$. That is $[C,B]$ is a subgroup of $(\prod_{\lambda\in I}[C_\lambda, B_\lambda])D[K,B]$. The reverse inclusion is trivial. //

To shorten the proof of the main theorem, a technical lemma is established.

LEMMA 2.2.4 $[C_\lambda, [B_\mu^{\ B}]]$ is a subgroup of D for all $\lambda \in I$.

Proof It suffices to prove that $[C_\lambda{}^B,[B_\mu{}^B]]$ is a subgroup of D since $[C_\lambda,[B_\mu{}^B]]$ is a subgroup of $[C_\lambda{}^B,[B_\mu{}^B]]$. The cartesian $[B_\mu{}^B]$ is generated by the elements $[b_\mu,b_\rho]$ where $b_\mu \in B_\mu{}^B$, $b_\rho \in B_\rho{}^B$; $\mu,\rho \in I$, $\mu \neq \rho$. Thus, because $C_\lambda{}^B$ is normal in B, $[C_\lambda{}^B,[B_\mu{}^B]]$ is generated by the elements $[b_\mu,b_\rho,c_\lambda]$, $b_\mu \in B_\mu{}^B$, $b_\rho \in B_\rho{}^B$, $c_\lambda \in C_\lambda{}^B$; $\mu \neq \rho$. Moreover,

$$[C_\lambda{}^B,B_\mu{}^B] = [C_\lambda,B_\mu{}^B]^B$$

$$\leq [C_\lambda,B_\lambda]D \qquad \text{by (2.2.3) (ii)}$$

Hence, $\quad [C_\lambda{}^B,B_\mu{}^B] \leq D \qquad \lambda \neq \mu, \quad$ by (2.1.3) .

The lemma is therefore proved if $[b_\mu,b_\rho,c_\lambda]$, $\mu \neq \rho$, can be shown to lie in $[C_\lambda{}^B,B_\sigma{}^B]$ for some $\sigma \neq \lambda$.

We have from (1.3.6) that,

$$[b_\mu,b_\rho,c_\lambda] = [c_\lambda,b_\mu{}^{-1},b_\rho{}^{-1}]^{-b_\mu b_\rho}\,[b_\rho{}^{-1},c_\lambda{}^{-1},b_\mu{}^{-1}]^{-c_\lambda b_\rho} \qquad - (*) .$$

Consider the first commutator. There are two cases. If $\lambda = \mu$, then $[c_\lambda,b_\mu{}^{-1},b_\rho{}^{-1}]$ lies in $[C_\lambda{}^B,B_\rho{}^B]$ since C_λ is normal in B_λ ($\lambda \neq \rho$ by hypothesis). If $\lambda \neq \mu$ then

$$[c_\lambda,b_\mu{}^{-1},b_\rho{}^{-1}] = [c_\lambda,b_\mu{}^{-1}]^{-1}[c_\lambda,b_\mu{}^{-1}]^{b_\rho{}^{-1}}$$

$$= [c_\lambda,b_\mu{}^{-1}]^{-1}[c_\lambda{}^{b_\rho{}^{-1}},b_\mu{}^{-b_\rho{}^{-1}}] \text{ which is in } [C_\lambda{}^B,B_\mu{}^B] .$$

That is $[c_\lambda,b_\mu{}^{-1},b_\rho{}^{-1}]^{-b_\mu b_\rho}$ is in the normal subgroup $[C_\lambda{}^B,B_\rho{}^B]$. The second commutator $[b_\rho{}^{-1},c_\lambda{}^{-1},b_\mu] = [[c_\lambda{}^{-1},b_\rho{}^{-1}]^{-1},b_\mu]$ can be treated in the same fashion.

The result follows from (*). //

THEOREM 2.2.5 $C\cap B'/[C,B]$ is isomorphic to the direct product $(\prod^{\times}_{\lambda\in I} (C_\lambda \cap B_\lambda')/[C_\lambda,B_\lambda]) \times H/[H,A]$.

Proof The argument is split into three parts.

(i) Let ϕ be the canonical homomorphism from B onto B/D. Then ϕ is a regular homomorphism since $\ker\phi = D$ and D is clearly a subgroup of $[B_\lambda{}^B]$. It follows from (2.1.7) that ϕ restricted to B_λ is an isomorphism for each $\lambda \in I$ and that $B\phi$ is a regular product of the $B_\lambda\phi$.

Consider the subgroup, $N = \mathrm{sgp}(C_\lambda\phi \mid \lambda \in I)K$. Now $[C_\lambda, C_\mu]$, $\lambda \neq \mu$, is a subgroup of $D = \ker\phi$ so that

$$[C_\lambda\phi, C_\mu\phi] = E \qquad \text{- (a)}$$

Also,

$$\begin{aligned}[C_\lambda, K] &\leq [C_\lambda, [B_\lambda{}^B]] && \text{since } K \leq [B_\lambda{}^B] \\ &\leq D && \text{by Lemma 2.2.4}.\end{aligned}$$

Thus

$$[C_\lambda\phi, K\phi] = E \qquad \text{- (b)}$$

In addition,

$$C_\lambda\phi \cap (\prod_{\substack{\mu\in I\\ \mu\neq\lambda}} C_\mu\phi)K \leq B_\lambda\phi \cap (\hat{B}_\mu\phi) = E \qquad \text{- (c)}$$

and

$$K\phi \cap \prod_{\lambda\in I} C_\lambda\phi \leq [B_\lambda\phi^{B\phi}] \cap \prod_{\lambda\in I} C_\lambda\phi = E \qquad \text{- (d)}$$

Conditions (a) to (d) are sufficient to make N a direct product of its subgroups $K\phi$ and $C_\lambda\phi$, $\lambda \in I$.

(ii) $$\begin{aligned} C\cap B'/[C,B] &\simeq \frac{C\cap B'/D}{[C,B]/D} && \text{since } D \leq [C,B] \text{ from (2.2.3)(ii)} \\ &= (C\cap B')\phi/[C,B]\phi . \end{aligned}$$

From (i) and (2.2.3), we have that $(C\cap B')\phi$ is the direct product of its subgroups $K\phi$ and $(C_\lambda \cap B_\lambda')\phi$, $\lambda \in I$, (since $D\phi = E$) and that $[C,B]\phi$ is the direct product of its subgroups $[K,B]\phi$ and $[C_\lambda,B_\lambda]\phi$, $\lambda \in I$. This, together with the facts that $[K,B]\phi \leq K\phi$ and $[C_\lambda,B_\lambda]\phi \leq (C_\lambda \cap B_\lambda')\phi$, implies that,

$$\begin{aligned}(C\cap B')\phi/[C,B]\phi &\simeq \Big(\prod_{\lambda\in I}^{\times} (C_\lambda\cap B_\lambda')\phi/[C_\lambda,B_\lambda]\phi\Big) \times K\phi/[K,B]\phi \\ &\simeq \Big(\prod_{\lambda\in I}^{\times} (C_\lambda\cap B_\lambda')/[C_\lambda,B_\lambda]\Big) \times K\phi/[K,B]\phi\end{aligned}$$

since ϕ is an isomorphism on B_λ.

(iii) In conclusion, $K\nu/[K,B]\nu$ is isomorphic to $K\phi/[K,B]\phi$ by (1.3.20) since

$$\begin{aligned}(K \cap \ker\nu)[K,B] &= (K \cap \prod_{\lambda\in I} C_\lambda^{B})[K,B] \\ &= (K \cap (\prod_{\lambda\in I} C_\lambda)D)[K,B] \\ &= (K \cap D)[K,B] \qquad \text{by (2.1.3) since } K,D \leq [B_\lambda{}^{B}] \\ &= (K \cap \ker\phi)[K,B] .\end{aligned}$$

But $K\nu/[K,B]\nu = H/[H,A]$ by construction. //

The main result is a corollary to Theorem 2.2.5.

THEOREM 2.2.6 $M(G)$ is isomorphic to the direct product $\big(\prod_{\lambda\in I}^{\times} M(A_\lambda)\big) \times H/[H,A]$ when G is a regular product of the A_λ, $A/H \simeq G$ and where $A = \prod_{\lambda\in I}^{*} A_\lambda$.

Proof If B_λ is a free group then B_λ/C_λ is a presentation for A_λ and $C_\lambda \cap B'/[C_\lambda,B_\lambda]$ equals $M(A_\lambda)$. The free product, B, will be a free group and, since B/C is isomorphic to G, $M(G)$ equals $C\cap B'/[C,B]$. The result follows immediately from Theorem 2.2.5. //

Theorem 2.2.6 reduces the problem of finding $M(G)$ to that of finding $H/[H,A]$. When $G = A_1 \times A_2$, $A = A_1 * A_2$ and

$$H/[H,A] = [A_1,A_2]/[[A_1,A_2],A] .$$

Wiegold ([29] Lemma 3.9) shows concisely that the latter group is isomorphic to $A_1 \otimes A_2$. So we have recovered Schur's decomposition of $M(A_1 \times A_2)$ referred to in the introduction. Going to the opposite extreme, H is trivial when $G = A_1 * A_2$ and hence $M(A_1 * A_2) = M(A_1) \times M(A_2)$. This is a particular case of the equation $H_n(A_1 * A_2,Z) = H_n(A_1,Z) \times H_n(A_2,Z)$, $n \geq 1$, due to Reinhart and Barr [1] 1, section 4.

The group $H/[H,A]$ has an interesting characterisation strongly analogous to the characterisation of $M(G)$, for an arbitrary finite group G, in terms of representing pairs (see Chapter 1, section 2).

Let $\bar{G}$ be a regular product of the A_λ such that G is the image of $\bar{G}$ under a regular homomorphism $\bar{\psi}$ (i.e.: $\bar{\psi}$ is the identity on each A_λ). Now, if $\bar{H} = \ker\bar{\psi}$ is also central in $\bar{G}$, then $\bar{G}$ may be properly called a "central regular extension of $\bar{H}$ by G". Consider all pairs $(\bar{G},\bar{H})$. We can interpret $H/[H,A]$ as a "maximal" second member of such a pair.

Because $[H,A]$ is contained in H, the epimorphism ψ from A to G induces an epimorphism ψ' from $A/[H,A]$ to G. But $[H,A]$ must also be contained in $[A_\lambda{}^A]$ so that $R(G) = A/[H,A]$ is a regular product of the A_λ and ψ' is a regular epimorphism of $R(G)$. The kernel of ψ' is $H/[H,A]$ which is central $R(G)$. Thus $(R(G),H/[H,A])$ is a pair of the above type.

The term maximal is applicable to $H/[H,A]$ in the sense that any other $\bar{H}$ is a homomorphic image of $H/[H,A]$. If η is the natural homomorphism from the free product A to $\bar{G}$ induced by the identity on each A_λ, then the following diagram commutes,

$$\begin{tikzcd} A \arrow[d, "\eta"'] \arrow[dr, "\psi"] & \\ \bar{G} \arrow[r, "\bar{\psi}"'] & G \end{tikzcd}$$

Thus $\bar{H} = \ker\bar{\psi} = H\eta$ and hence $[H,A]\eta = [\bar{H},\bar{G}] = E$. It follows that η induces a homomorphism η' from $R(G)$ to $\bar{G}$ where $(H/[H,A])\eta' = H\eta = \bar{H}$.

Theorem 2.2.6 implies that $M(G)$ is trivial if and only if $H/[H,A]$ and $M(A_\lambda)$, $\lambda \in I$, are trivial. It would therefore be worth finding conditions on H for which $H = [H,A]$.

We examine the structure of $H/[H,A]$ in detail for the class of nilpotent products in Chapter 3.

3. Construction of a representing group

If G is finite then it must be generated by a finite set of subgroups $A_1, A_2, \ldots, A_n$. Let $(L_i, M(A_i))$ be a representing pair for A_i, $i = 1, 2, \ldots, n$. Then these can be used to form a representing pair for G when G is a regular product.

We use the same construction as before. There is an epimorphism from L_i to A_i with kernel $M(A_i)$ by hypothesis. Let $L = \prod_{i=1}^{n}{}^{*} L_i$ and Φ be the natural epimorphism from L onto A induced by these epimorphisms. Put $J = H_1 \cap [L_i^{\,L}]$, where H_1 is the group in L which maps onto H under σ, and let $N = \prod_{\substack{i,j=1 \\ i\neq j}}^{n} [M(A_i), L_j]^L$. Then $J\sigma = H$ is normal in L.

THEOREM 2.3.1 Suppose G is finite and a regular product of its subgroups $A_1, A_2, \ldots, A_n$. Let $(L_i, M(A_i))$ be a fixed representing pair for A_i. Then $(L\tau, M\tau)$ is a representing for for G where

$\tau : L \to L/N[J,L]$ and $M = (\prod_{i=1}^{n} M(A_i))J$ in L. The groups L, N and J are as defined above.

Proof (i) Apply Lemma 2.2.1 by putting $L = \{1, 2, \ldots, n\}$, $B_i = L_i$ and $C_i = M(A_i)$. Then $L = B$, $C = M$, $N = D$ and $J = K$. We have,

$$G \simeq B/C$$

$$\simeq \frac{B/D[K,B]}{C/D[K,B]} \quad \text{since } D[K,B] \leq C \text{ from (2.2.3)(i)}.$$

Thus, $$G = \frac{L/N[J,L]}{M/N[J,L]} = L\tau/M\tau \quad \text{by definition of } \tau .$$

Now J is a subgroup of L' immediately from its definition, and $M(A_i)$ is a subgroup of L_i' by hypothesis, so that M is a subgroup of L'. That is $(M\tau)'$ is a subgroup of $(L\tau)'$.

The kernel of τ contains $[J,L]$ as a subgroup which means that $J\tau$ is central in $L\tau$. Also $[M(A_i),L_j]$, $i \neq j$, is a subgroup of N which is in $\ker\tau$. This $M(A_i)\tau$ commutes with $L_j\tau$ for $i \neq j$. If $i = j$ then $M(A_i)$ commutes with L_i by hypothesis. Hence $M\tau$, as a product of $J\tau$ and the $M(A_i)\tau$, is central in $L\tau$.

$(L\tau,M\tau)$ is therefore a defining pair and will be a representing pair if $M\tau$ is isomorphic to $M(G)$ (see section 2).

(ii) Theorem 2.2.5 gives that

$$M\cap L'/[M,L] \simeq (\prod_{i=1}^{n}{}^{\times} (M(A_i) \cap L_i')/[M(A_i),L_i]) \times H/[H,A] .$$

But $M(A_i) \leq L_i' \cap Z(L_i)$ so that $M(A_i) \cap L_i' = M(A_i)$ and $[M(A_i),L_i] = E$. Hence

$$M\cap L'/[M,L] \simeq (\prod_{i=1}^{n} M(A_i)) \times H/[H,A]$$

$$\simeq M(G) \qquad \text{by Theorem 2.2.6}.$$

Looking at the quotient $M \cap L'/[M,L]$, we have from 2.2.3(ii) that,

$$[M,L] = \Big(\prod_{\substack{i,j=1 \\ i \neq j}}^{n} [M(A_i),L_j] \Big)[J,L]$$

$$= N[J,L]$$

$$= \ker\tau \; .$$

Thus, $$M\tau = M \cap L'/[M,L] \simeq M(G) \; . \qquad //$$

The kernel τ is clearly a subgroup of $[L_i{}^L]$ so that $L\tau$ is a regular product of the L_i.

Wiegold has proved a special case of Theorem 2.3.1 in [32] using a different approach. He shows that the second nilpotent product $L_1(2)L_2$ is a representing group for $A_1 \times A_2$: a fact which can be deduced from Theorem 2.3.1 via Theorem 3.1.8.

CHAPTER 3

NILPOTENT PRODUCTS

1. General results

Suppose $G \simeq A/H$, where $A = \prod^*_{\lambda \in I} A_\lambda$, is a regular product. Denote $H/[H,A]$ by $MG(A_\lambda)$. We recall from Theorem 2.2.6 that, given the $M(A_\lambda)$, calculating $M(G)$ amounts to finding $MG(A_\lambda)$. This seems easiest for the class of nilpotent products. The m^{th} nilpotent product of the A_λ can be shown to be the regular product, G, with H equal to the $(m+1)^{th}$ term of the series, $[A_\lambda{}^A] \geq [[A_\lambda{}^A],A] \geq [[[A_\lambda{}^A],A],A] \geq \ldots$. It follows that, if G_m is the m^{th} nilpotent product of the A_λ, then $MG_m(A_\lambda)$ is a central subgroup of G_{m+1}. So the calculation remains confined to the fairly well known class of nilpotent products; whereas, in general, $MG(A_\lambda)$ is a central subgroup of the regular product $R(G) = A/[H,A]$ whose structure may be quite different from G.

In this section we establish an improved version of the representing group construction of Theorem 2.3.1 for $G = G_m$ and prove two reduction theorems which simplify $MG_m(A_\lambda)$.

It is most convenient to start by introducing arbitrary verbal products and defining nilpotent products as a special case. Verbal products will be needed again in Chapter 4.

The notion of a verbal subgroup is covered fully in Neumann [21].

DEFINITION 3.1.1 ([21] 12.21) Let F_∞ be the free group of countable rank. If K is an arbitrary group and V a subset of F_∞, then the verbal subgroup of K corresponding to V is the group,

$$V(K) = \operatorname{sgp}(v\alpha \mid v \in V,\ \alpha \in \operatorname{Hom}(F_\infty, K)) .$$

It is easy to check from the definition that,

(3.1.2) If ϕ is a homomorphism of K then $V(K)\phi = V(K\phi)$. In particular, $V(K)$ is normal in K.

This leads to Moran's definition of a verbal product.

DEFINITION 3.1.3 ([18] 4.1) Suppose $V(A)$ is a verbal subgroup of $A = \prod^*_{\lambda\in I} A_\lambda$. Then the V-verbal product of the A_λ, $\lambda \in I$, is the group,

$$\prod^V_{\lambda\in I} A_\lambda = A/(V(A) \cap [A_\lambda{}^A]) .$$

The product $\prod^V_{\lambda\in I} A_\lambda$ is regular by (2.1.8) and may therefore be considered as being generated by the A_λ themselves. Moran has proved that all verbal multiplications are associative [18] section 5.

We digress to state a trivial extension of Theorem 3.5 in [18] which will be used frequently in both this chapter and the next.

LEMMA 3.1.4 Suppose ϕ is a homomorphism of $A = \prod^*_{\lambda\in I} A_\lambda$. Then $(V(A) \cap [A_\lambda{}^A])\phi = V(A\phi) \cap [A_\lambda\phi^{A\phi}]$ if either,

(i) ϕ is regular

or, (ii) There exists,

(a) a mapping Γ of the index set I

(b) a collection of groups B_μ, $\mu \in I\Gamma$

(c) homomorphisms $\alpha_\lambda : A_\lambda \to B_{\lambda\Gamma}$ for each $\lambda \in I$ such that ϕ restricted to A_λ is α_λ.

Returning to the development of nilpotent products, suppose that F is freely generated by the elements $x_1, x_2, x_3, \ldots$. It is usual to denote V(K) by $\gamma_m(K)$ when V consists of the single commutator $[x_1, x_2, \ldots, x_m]$, $m \geq 2$. Putting $\gamma_1(K) = K$ implies that $\gamma_m(K) = [\gamma_{m-1}(K), K]$, for $m \geq 2$, by induction on m. Since $\gamma_m(K)$ is normal in K (by (3.1.2)), $[\gamma_m(K), K]$ is a normal subgroup of $\gamma_m(K)$ and we have a chain of normal subgroups,

$$K = \gamma_1(K) \geq \gamma_2(K) \geq \gamma_3(K) \geq \ldots$$

called the <u>lower central series of K</u>. The group K is said to be <u>nilpotent of class m</u> when $\gamma_{m+1}(K) = E$, or equivalently, when $\gamma_m(K)$ is central in K. Nilpotent products are defined in an analogous way.

DEFINITION 3.1.5 ([19] section 8) The m^{th} nilpotent product of the A_λ, $\lambda \in I$, for $m \geq 1$, is the group,

$$\prod_{\lambda \in I}^{(m)} A_\lambda = A/C_m(A_\lambda)$$

where $A = \prod^*_{\lambda \in I} A_\lambda$ and $C_m(A_\lambda) = \gamma_{m+1}(A) \cap [A_\lambda{}^A]$.

This is just definition 3.1.3 with $V(A) = \gamma_{m+1}(A)$. Note that $\prod^{(1)}_{\lambda \in I} A_\lambda = \prod^{\times}_{\lambda \in I} A_\lambda$.

We will need the following result on the lower central series of a regular product.

LEMMA 3.1.6 Suppose G is a regular product of the A_λ, $\lambda \in I$. Then, $\gamma_m(G) = (\prod_{\lambda \in I} \gamma_m(A_\lambda))(\gamma_m(G) \cap [A_\lambda{}^G])$.

Now,

$$MG_m(A_\lambda) = C_{m+1}(A_\lambda)/[C_{m+1}(A_\lambda), A] , \qquad m \geq 1 .$$

However,

$$(3.1.7)\quad [C_{m+1}(A_\lambda),A] = C_{m+2}(A_\lambda), \qquad \text{for } m \geq 1 .$$

Proof Define the descending series of normal subgroups ${}_m[A_\lambda]$, $m \geq 0$, recursively by putting ${}_0[A_\lambda] = [A_\lambda{}^A]$ and

$${}_m[A_\lambda] = [{}_{(m-1)}[A_\lambda],A], \qquad \text{for } m \geq 1 .$$

Note that ${}_m[A_\lambda]$ is a subgroup of $\gamma_{m+2}(A)$ by induction on m.

Suppose ϕ is the canonical homomorphism from A onto $A/{}_m[A_\lambda]$. Then, from [6] Chapter II, corollary 6.9,

$$\gamma_{m+2}(A\phi) = \prod_{\lambda\in I} \gamma_{m+2}(A_\lambda\phi) \qquad \text{for } m \geq 0$$

Hence,

$$\gamma_{m+2}(A) = \Big(\prod_{\lambda\in I} \gamma_{m+2}(A_\lambda)\Big)\,{}_m[A_\lambda] \qquad \text{by (3.1.2)}$$

But,

$$\gamma_{m+2}(A_\lambda) = \Big(\prod_{\lambda\in I} \gamma_{m+2}(A_\lambda)\Big)C_{m+2}(A_\lambda) \qquad \text{by (3.1.6)}$$

Thus,

$$C_{m+2}(A_\lambda) = {}_m[A_\lambda], \qquad \text{for } m \geq 0, \text{ by the uniqueness result (2.1.3).}$$

It follows that,

$$\begin{aligned} [C_{m+1}(A_\lambda),A] &= [{}_{(m-1)}[A_\lambda],A] \\ &= {}_m[A_\lambda] \\ &= C_{m+2}[A_\lambda] \qquad \text{for } m \geq 1 . \end{aligned}$$

//

Consequently $MG_m(A_\lambda)$ can be identified with a subgroup of G_{m+1}.

THEOREM 3.1.8 $MG_m(A_\lambda) = \gamma_{m+1}(G_{m+1}) \cap [A_\lambda{}^{G_{m+1}}]$ for $m \geq 1$ where

$G_m = \prod^{(m)}_{\lambda\in I} A_\lambda$. This makes $MG_m(A_\lambda)$ a central subgroup of G_{m+1}.

Proof Let ψ be the natural homomorphism from A onto G_{m+1} induced by the identity on each A_λ. Then,

$$\begin{aligned}
MG_m(A_\lambda) &= C_{m+1}(A_\lambda)/[C_{m+1}(A_\lambda),A] \\
&= C_{m+1}(A_\lambda)/C_{m+2}(A_\lambda) \quad \text{by (3.1.7)} \\
&= C_{m+1}(A_\lambda)\psi \\
&= (\gamma_{m+1}(A) \cap [A_\lambda{}^A])\psi \\
&= \gamma_{m+1}(A\psi) \cap [A_\lambda\psi^{A\psi}] \quad \text{by Lemma (3.1.4) since } \psi \text{ is regular} \\
&= \gamma_{m+1}(G_{m+1}) \cap [A_\lambda{}^{G_{m+1}}] \ .
\end{aligned}$$

Moreover,

$$\begin{aligned}
[MG_m(A_\lambda),G_{m+1}] &= [C_{m+1}(A_\lambda)\psi,A\psi] \\
&= [C_{m+1}(A_\lambda),A]\psi \\
&= C_{m+2}(A_\lambda)\psi \\
&= E
\end{aligned}$$

and $MG_m(A_\lambda)$ is central. //

Proposition (3.1.7) also yields the promised construction of a representing group for G_m. The free product L of representing groups L_i, i = 1, 2, ..., n in Theorem 2.3.1 can be replaced by their $(m+1)^{th}$ nilpotent product.

THEOREM 3.1.8 Suppose $G_m = \prod^{(m)n}_{i=1} A_i$, $m \geq 1$, is finite and $(L_i,M(A_i))$ is a representing pair for A_i; i = 1, 2, ... n. Then $(\bar{L}\bar{\tau},\bar{M}\bar{\tau})$ is a representing pair for G_m where

(i) $\bar{L} = \prod_{i=1}^{n}{}^{(m+1)} L_i$

(ii) $\bar{M} = (\prod_{i=1}^{n} M(A_i))(\gamma_{m+1}(\bar{L}) \cap [L_i{}^{\bar{L}}])$ in $\bar{L}$

(iii) $\bar{\tau} = \bar{L} \to \bar{L}/\bar{N}$

(iv) $\bar{N} = \prod_{\substack{i,j=1 \\ i \neq j}}^{n} [M(A_i), L_j]^{\bar{L}}$ in $\bar{L}$.

Proof Let L, N, J, M, τ and σ be defined as in Theorem 2.3.1. We construct an isomorphism $\chi : L\tau \to \bar{L}\bar{\tau}$ which maps $M\tau$ onto $\bar{M}\bar{\tau}$ and the result follows straight from Theorem 2.3.1. The trick is to show that $\ker\tau = \ker\eta\bar{\tau}$ where η is the natural homomorphism from L to $\bar{L}$ induced by the identity on each L_i.

Now,

$$\begin{aligned} C_{m+1}(L_i)\sigma &= (\gamma_{m+1}(L) \cap [L_i{}^{L}])\sigma \\ &= \gamma_{m+1}(A) \cap [A_i{}^{A}] && \text{by Lemma 3.1.4 since } \sigma : L \to A \text{ takes } L_i \text{ onto } A_i \\ &= C_{m+1}(A_i) \\ &= H && \text{for } G = G_{m+1} \text{ by definition .} \end{aligned}$$

Thus,

$$\begin{aligned} H_1 &= \ker\sigma\, C_{m+1}(L_i) && \text{since } H_1 \text{ is the preimage of } H \text{ under } \sigma \\ &= (\prod_{i=1}^{n} M(A_i)^{L})C_{m+1}(L_i) && \text{by Lemma 2.2.1} \end{aligned}$$

and,

$$J = H_1 \cap [L_i{}^L]$$

$$= (\prod_{i=1}^{n} M(A_i)^L)C_{m+1}(L_i) \cap [L_i{}^L]$$

$$= (\prod_{i=1}^{n} M(A_i)^L \cap [L_i{}^L])C_{m+1}(L_i) \qquad \text{since } C_{m+1}(L_i) \leq [L_i{}^L]$$

$$= ((\prod_{i=1}^{n} M(A_i))N \cap [L_i{}^L])C_{m+1}(L_i) \qquad \text{by Lemma 2.2.2}$$

$$= N\,C_{m+1}(L_i) \qquad \text{by (2.1.3) since } N \leq [L_i{}^L] \,.$$

It follows that,

$$\ker\tau = N[J,L]$$

$$= N[N\,C_{m+1}(L_i),L]$$

$$= N[N,L][C_{m+1}(L_i),L] \qquad \text{by (1.3.17)}$$

$$= N\,C_{m+2}(L_i) \qquad \text{by (3.1.7) and the normality of } N\,.$$

But $N\eta = \bar{N}$ and $\bar{N} = \ker\bar{\tau}$ so that,

$$\ker\eta\bar{\tau} = N\ker\eta$$

$$= N\,C_{m+2}(L_i)$$

$$= \ker\tau\,.$$

Hence the map $\chi : x\tau \mapsto (x\eta)\bar{\tau}$, $x \in L$, is an isomorphism from $L\tau$ onto $\bar{L}\bar{\tau}$. Also,

$$(M\tau)\chi = ((\prod_{i=1}^{n} M(A_i))(\gamma_{m+1}(L) \cap [L_i{}^L]))\eta\bar{\tau}$$

$$= ((\prod_{i=1}^{n} M(A_i))(\gamma_{m+1}(\bar{L}) \cap [L_i{}^{\bar{L}}]))\bar{\tau} \qquad \text{by Lemma 3.1.4 since } \eta \text{ is regular}$$

$$= \bar{M}\bar{\tau}\,. \qquad //$$

The first reduction theorem below says in effect that, in calculating $MG_m(A_\lambda)$, the factors of G_m may be considered as being nilpotent, class m.

THEOREM 3.1.10 Suppose $G_m = \prod_{\lambda\in I}^{(m)} A_\lambda$ and $\bar{A}_\lambda = A_\lambda/\gamma_{m+1}(A_\lambda)$, then

$$MG_m(A_\lambda) \simeq M\bar{G}_m(\bar{A}_\lambda) \quad \text{where } \bar{G}_m = \prod_{\lambda\in I}^{(m)} \bar{A}_\lambda .$$

Proof Let α be the natural homomorphism from A to $\prod^*_{\lambda\in I} \bar{A}_\lambda$ induced by the canonical homomorphisms from each A_λ to $\bar{A}_\lambda$, $\lambda \in I$. Then,

$$M\bar{G}_m(\bar{A}_\lambda) = C_{m+1}(\bar{A}_\lambda)/C_{m+2}(\bar{A}_\lambda)$$

$$= C_{m+1}(A_\lambda)\alpha/C_{m+2}(A_\lambda)\alpha \quad \text{by Lemma 3.1.4(ii)} .$$

If it can be shown that $(C_{m+1}(A_\lambda) \cap \ker\alpha) \leq C_{m+2}(A_\lambda)$, (i.e.: $(C_{m+1}(A_\lambda) \cap \ker\alpha)C_{m+2}(A_\lambda) = C_{m+2}(A_\lambda)$) then the result will follow from (1.3.20) since

$$MG_m(A_\lambda) = C_{m+1}(A_\lambda)/C_{m+2}(A_\lambda) .$$

Now,

$$C_{m+1}(A_\lambda) \cap \ker\alpha = C_{m+1}(A_\lambda) \cap \prod_{\lambda\in I} \gamma_{m+1}(A_\lambda)^A \quad \text{by Lemma 2.2.1}$$

$$= C_{m+1}(A_\lambda) \cap (\prod_{\lambda\in I} \gamma_{m+1}(A_\lambda))D \quad \text{where}$$

$$D = \prod_{\substack{\lambda,\mu\in I\\ \lambda\neq\mu}} [\gamma_{m+1}(A_\lambda),A_\mu]$$

$$= C_{m+1}(A_\lambda)\cap D \quad \text{by (2.1.3) since}$$

$$D, C_{m+1}(A_\lambda) \leq [A_\lambda{}^A] .$$

But $[\gamma_{m+1}(A_\lambda),A_\mu]$, $\lambda \neq \mu$, is a subgroup of $[A_\lambda,A_\mu]$ which is, in turn, contained in $[A_\lambda{}^A]$. Also $[\gamma_{m+1}(A_\lambda),A_\mu]$ is in $\gamma_{m+2}(A)$. This means that $[\gamma_{m+1}(A_\lambda), A_\mu]$ is a subgroup of $\gamma_{m+2}(A) \cap [A_\lambda{}^A] = C_{m+2}(A_\lambda)$. Thus D, and hence $C_{m+1}(A_\lambda)$ D, is a subgroup of $C_{m+2}(A_\lambda)$. //

Theorem 3.1.10 can be strengthened in the case where the $\bar{A}_\lambda$ are finite by using the well known result that a finite nilpotent group is the direct product of its Sylow subgroups. Two preliminary results are necessary.

(3.1.11) ([6] Chapter II, lemma 6.5) If $a_1 \in A_1$, $a_2 \in A_2$ have relatively prime orders, then the commutator $[a_1, a_2]$ evaluated in $A_1(m)A_2$ is trivial for all $m \geq 1$.

LEMMA 3.1.12 Suppose that A_λ is periodic and that P_λ is the set of primes which divide the orders of the elements of A_λ, $\lambda \in I$. If $P_\lambda \cap P_\mu$ is empty for each $\lambda \neq \mu$ then $\prod^{(m)}_{\lambda\in I} A_\lambda = \prod^{\times}_{\lambda \in I} A_\lambda$ for all $m \geq 1$.

Proof The associativity of nilpotent multiplication implies that $\mathrm{sgp}(A_\lambda, A_\mu) = A_\lambda(m)A_\mu$ in G_m, $\lambda \neq \mu$. But if $a_\lambda \in A_\lambda$, $a_\mu \in A_\mu$, then a_λ and a_μ have relatively prime orders by hypothesis. Thus $[a_\lambda, a_\mu] = 1$ and hence $[A_\lambda, A_\mu] = E$ by (3.1.11). Moreover,

$$A_\lambda \cap \mathrm{sgp}(A_\mu | \mu \in I, \mu \neq \lambda) \leq A_\lambda \cap \hat{A}_\lambda = E .$$

These conditions are sufficient to make G_m a direct product of the A_λ. //

THEOREM 3.1.13 Suppose $G_m = \prod^{(m)}_{\lambda\in I} A_\lambda$ and each $\bar{A}_\lambda = A_\lambda/\gamma_{m+1}(A_\lambda)$ is finite. Let P be the set of primes dividing $|\bar{A}_\lambda|$, $\lambda \in I$, and further, let $S_{p,\lambda}$ be the Sylow p-subgroup of $\bar{A}_\lambda$ for $p \in P$. Then

$$MG_m(A_\lambda) \simeq \prod^{\times}_{p\in P} MG_{p,m}(S_{p,\lambda})$$

where $G_{p,m} = \prod_{\lambda\in I}^{(m)} S_{p,\lambda}$ for each $p \in P$.

Proof The proof relies on the fact that $M\bar{G}_m(\bar{A}_\lambda)$ is a subgroup of the cartesian of $\bar{G}_{m+1} = \prod_{\lambda\in I}^{(m+1)} \bar{A}_\lambda$ (Theorem 3.1.8).

(i) Note that, for fixed λ, all but a finite number of the $S_{p,\lambda}$ are trivial.

$\bar{G}_{m+1}$	$= \prod_{\lambda\in I}^{(m+1)} (\prod_{p\in P}^{\times} S_{p,\lambda})$	since $\bar{A}_\lambda$ is the direct product of its Sylow subgroups
	$= \prod_{\lambda\in I}^{(m+1)} (\prod_{p\in P}^{(m+1)} S_{p,\lambda})$	by Lemma 3.1.12
	$= \prod_{\substack{\lambda\in I\\ p\in P}}^{(m+1)} S_{p,\lambda}$	by the associativity and commutativity of nilpotent products
	$= \prod_{p\in P}^{(m+1)} G_{p,(m+1)}$	similarly
	$= \prod_{p\in P}^{\times} G_{p,(m+1)}$	by Lemma 3.1.12 and [6] Chapter II, corollary 6.7 - which states that $G_{p,(m+1)}$ is a p-group.

(ii) Certainly $\mathrm{sgp}([S_{p,\lambda}{}^{G_{p,(m+1)}}] \mid p \in P)$ is contained in $[\bar{A}_\lambda{}^{\bar{G}_{m+1}}]$. We prove the opposite inclusion.

Consider elements $s_{p,\lambda} \in S_{p,\lambda}$ and $r_{p,\lambda} \in \mathrm{sgp}(s_{q,\lambda} \mid q \in P, q \neq p)$ in $\bar{G}_{m+1}$. Then by the commutator identities (1.3.4) and (1.3.5),

$$[s_{p,\lambda}r_{p,\lambda}, s_{p,\mu}r_{p,\mu}]$$

$$= [s_{p,\lambda}, r_{p,\mu}]^{r_{p,\lambda}}[s_{p,\lambda}, s_{p,\mu}]^{r_{p,\mu}r_{p,\lambda}}[r_{p,\lambda}, r_{p,\mu}][r_{p,\lambda}, s_{p,\mu}]^{r_{p,\mu}}.$$

But the elements $s_{p,\lambda}$, $r_{p,\rho}$ commute for all $\lambda, \rho \in I$ by (i) so that,

$$[s_{p,\lambda}r_{p,\lambda}, s_{p,\mu}r_{p,\mu}] = [s_{p,\lambda}, s_{p,\mu}][r_{p,\lambda}, r_{p,\mu}] \quad - \quad (*)$$

The cartesian $[\bar{A}_\lambda{}^{\bar{G}_{m+1}}]$ is generated by the elements $[x_\lambda, x_\mu]^x$ where $x_\lambda \in \bar{A}_\lambda$, $x_\mu \in \bar{A}_\mu$, $x \in \bar{G}_{m+1}$. Since $\bar{A}_\lambda = \prod^{\times}_{p\in P} S_{p,\lambda}$ and $G_{m+1} = \prod^{\times}_{p\in P} G_{p,(m+1)}$, it is possible to express given x_λ, x_μ, x in A_λ, A_μ and $\bar{G}_{m+1}$ as,

$$x_\lambda = s_{p_1,\lambda} s_{p_2,\lambda} \cdots s_{p_\ell,\lambda}$$

$$x_\mu = s_{p_1,\mu} s_{p_2,\mu} \cdots s_{p_\ell,\mu}$$

$$x = g_{p_1} g_{p_2} \cdots g_{p_\ell}$$

where $s_{p_i,\lambda} \in S_{p_i,\lambda}$, $s_{p_i,\mu} \in S_{p_i,\mu}$, $g_{p_i} \in G_{p_i,(m+1)}$; $i = 1, 2, \ldots \ell$; $p_1, p_2, \ldots, p_\ell$ are distinct elements of P and some of the $s_{p_i,\lambda}$, $s_{p_i,\mu}$ and g_{p_i} are possibly zero. An application of (*) ℓ times to $[x_\lambda, x_\mu]$ yields,

$$[x_\lambda, x_\mu] = [s_{p_1,\lambda}, s_{p_1,\mu}][s_{p_2,\lambda}, s_{p_2,\mu}] \cdots [s_{p_\ell,\lambda}, s_{p_\ell,\mu}] .$$

Now g_{p_i} commutes with $[s_{p_j,\lambda}, s_{p_j,\mu}]$ for $i \neq j$ by (i). Hence,

$$[x_\lambda, x_\mu]^x = [s_{p_1,\lambda}, s_{p_1,\mu}]^{g_{p_1}} [s_{p_2,\lambda}, s_{p_2,\mu}]^{g_{p_2}} \cdots [s_{p_\ell,\lambda}, s_{p_\ell,\mu}]^{g_{p_\ell}}$$

and $[x_\lambda, x_\mu]^x$ is an element of $\mathrm{sgp}([S_{p\lambda}{}^{G_{p,(m+1)}}] \mid p \in P)$.

It follows from (i) that,

$$[\bar{A}_\lambda{}^{\bar{G}_{m+1}}] = \prod^{\times}_{p\in P} [S_{p,\lambda}{}^{G_{p,(m+1)}}] .$$

(iii) Finally [18] lemma 3.3 shows that $\gamma_{m+1}(\bar{G}_{m+1}) = \prod^{\times}_{p\in P} \gamma_{m+1}(G_{p,(m+1)})$. Thus,

$$MG_m(A_\lambda) \simeq M\bar{G}_m(\bar{A}_\lambda) \qquad \text{by Theorem 3.1.10}$$

$$= \gamma_{m+1}(\bar{G}_{m+1}) \cap [\bar{A}_\lambda{}^{\bar{G}_{m+1}}] \qquad \text{by Theorem 3.1.8}$$

$$= \prod_{p\in P}^{\times} \gamma_{m+1}(G_{p,(m+1)}) \cap \prod_{p\in P}^{\times} [S_{p,\lambda}{}^{G_{p,(m+1)}}] \quad \text{from (ii)}$$

$$= \prod_{p\in P}^{\times} (\gamma_{m+1}(G_{p,(m+1)}) \cap [S_{p,\lambda}{}^{G_{p,(m+1)}}])$$

$$= \prod_{p\in P}^{\times} MG_{p,m}(S_{p,\lambda}) \quad \text{again from Theorem 3.1.8 .//}$$

In particular, finding $M(G_m)$ for G_m finite reduces to finding $M(G_m)$ for G_m a finite p-group.

2. <u>Calculation of a basis for $MG_m(A_i)$, G_m finite</u>

When $G_m = \prod_{i=1}^{n}{}^{(m)} A_i$ is finite, $MG_m(A_i)$ is finite abelian. Furthermore, if $m = 1$, $n = 2$, then $MG_1(A_i) = A_1 \otimes A_2$ and this has a basis consisting of the elements $a_i \otimes b_j$, $i = 1, 2, \ldots \ell$; $j = 1, 2, \ldots, \ell'$; where the $a_i\gamma_2(A_1)$ and $b_j\gamma_2(A_2)$ form a basis for $A_1/\gamma_2(A_1)$ and $A_2/\gamma_2(A_2)$ respectively. One would hope, on the strength of this, that a basis could be found for $MG_m(A_i)$ where $m > 1$, $n = 2$, in terms of bases for $\gamma_k(A_1)/\gamma_{k+1}(A_1)$ and $\gamma_k(A_2)/\gamma_{k+1}(A_2)$, $1 \leq k \leq m$. However, it appears that knowing just these bases is insufficient information to produce a general determination of a basis for $MG_m(A_i)$. We examine a method of calculating the required basis and give three examples (for $m = 2$) which show that variations in the structure of A_1 and A_2, which preserve the structure of the factors of their lower central series, can cause large and perhaps unexpected changes in $MG_m(A_i)$. The general problem of relating a basis of $MG_m(A_i)$, $i = 1,2$ to the structures of A_1 and A_2 remains unsolved.

Suppose firstly that $A_1 = F_1$ and $A_2 = F_2$ where F_1 and F_2 are groups freely generated by $x_1, x_2, \ldots, x_\ell$ and $y_1, y_2, \ldots, y_{\ell'}$, respectively. Now, if α_m is the canonical homomorphism from $F = F_1 * F_2$ onto

$F/\gamma_{m+2}(F)$ then,

$$MG_m(F_i) \simeq C_{m+1}(F_i)\alpha_m \quad \text{by (1.3.20).}$$

This makes $MG_m(F_i)$ a subgroup of the quotient, $\gamma_{m+1}(F)/\gamma_{m+2}(F)$, of free groups for which a basis can be found in terms of basic commutators. In fact,

$$\gamma_{m+1}(F) = \gamma_{m+1}(F_1)\gamma_{m+1}(F_2)C_{m+1}(F_i) \quad \text{from (3.1.6).}$$

Therefore, by (2.1.3)

$$\gamma_{m+1}(F)\alpha_m = \gamma_{m+1}(F_1)\alpha_m \times \gamma_{m+1}(F_2)\alpha_m \times C_{m+1}(F_i)\alpha_m .$$

Our basis for $\gamma_{m+1}(F)\alpha_m$ will consist of commutators (modulo $\gamma_{m+2}(F)$) with entries of elements purely in F_1, purely in F_2 and mixed in F_1 and F_2. The latter commutators will form a basis for $C_{m+2}(F_i)\alpha_m$.

DEFINITION 3.2.1 A basic sequence on the symbols $c_1, c_2, \ldots, c_\ell$ is defined recursively in terms of formal bracket operations.

(i) Let $c_1, c_2, \ldots, c_\ell$ be the basic symbols of weight one.

(ii) Suppose that the basic symbols of weight less than m have been defined and that there are finitely many of them. Suppose also that they have been numbered $c_1, c_2, \ldots, c_r$. Then the basic symbols of weight m are the formal bracket symbols (c_i, c_j) such that,

(a) $wtc_i + wtc_j = m$ where wtc_i denotes the weight of c_i.

(b) $i > j$.

(c) if $c_i = (c_s, c_t)$ then $j \geq t$.

There are finitely many such pairs (at most r^2). Number them $c_{r+1}, c_{r+2}, \ldots$ in any fixed manner.

It is usual to order the symbols according to the rule, $c_i > c_j$ if and only if $i > j$.

If $c_1, c_2, \ldots, c_\ell$ are replaced in order by the free generators $x_1, x_2, \ldots x_\ell$ in F_1 and the bracket symbols (c_i, c_j) are replaced inductively by the commutators $[c_i, c_j]$ in F_1, then the sequence of elements $c_1, c_2, c_3, \ldots$ of F_1 is called a sequence of basic commutators on $x_1, x_2, \ldots, x_\ell$. Marshall Hall [9] has proved, using P. Hall's collecting process [11], that the c_i are distinct elements of F_1 and that,

THEOREM 3.2.2 The quotient $\gamma_m(F_1)/\gamma_{m+1}(F_1)$ is a free abelian group with a basis consisting of the cosets $c_i\gamma_{m+1}(F_1)$ where c_i is a basic commutator of weight m.

For example, if we take $x_1, x_2, \ldots, x_\ell, y_1, y_2, \ldots, y_{\ell'}$ with the ordering $x_1 < x_2 < \ldots < x_\ell < y_1 < y_2 < \ldots < y_{\ell'}$ as basic commutators of weight one in F, then $\gamma_3(F)/\gamma_4(F)$ has the commutators,

$$\left.\begin{array}{ll}
[x_i, x_j, x_k] & j \leq k,\ i > j \\
[x_i, x_j, y_r] & i > j \\
[y_r, x_i, x_j] & i \leq j \\
[y_r, x_i, y_s] & \\
[y_r, y_s, y_t] & s \leq t,\ r > s
\end{array}\right\} \text{ modulo } \gamma_4(F)$$

as a basis where $i, j = 1, 2, \ldots, \ell$; $r, s = 1, 2, \ldots, \ell'$. Clearly the commutators of the form,

$$\left.\begin{array}{ll} [x_i,x_j,y_r]\alpha_2 & i > j \\ [y_r,x_i,x_j]\alpha_2 & i \le j \\ [y_r,x_i,y_s]\alpha_2 & \end{array}\right\} \quad - (1)$$

are a basis for $C_3(F_i)\alpha_2$. It is a straightforward adaption of the work of Section 3 to prove that, if $N(\ell,m)$ is the number of basic commutators of weight m on ℓ elements (of weight one), the $C_{m+1}(F_i)\alpha_2$ has a basis consisting of $N(\ell+\ell',m+1) - N(\ell,m+1) - N(\ell',m+1)$ elements.

Suppose now that ν_1 and ν_2 are epimorphisms from F_1 and F_2 onto A_1 and A_2 with kernels R_1 and R_2. Let ν be the natural homomorphsim from F to $G_m = A_1(m)A_2$ induced by ν_1 and ν_2. Then,

$$MG_m(A_i) \;=\; C_{m+1}(F_i)\nu/C_{m+2}(F_i)\nu \quad \text{by Lemma 3.1.4 .}$$

But $\ker\nu = R_1R_2D$ where $D = [R_1,F_2]^F[R_2,F_1]^F$ by Lemma 2.2.1 and Lemma 2.2.2. Thus,

$$\begin{aligned}(C_{m+1}(F_i) \cap \ker\nu)C_{m+2}(F_i) &= (\gamma_{m+1}(F) \cap [F_i{}^F] \cap R_1R_2D)C_{m+2}(F_i) \\ &= (\gamma_{m+1}(F) \cap D)C_{m+2}(F_i) \qquad \text{by (2.1.3)} \\ &= (\gamma_{m+1}(F) \cap D)(C_{m+1}(F_i) \cap \gamma_{m+2}(F))\end{aligned}$$

which implies that,

$$MG_m(A_i) \;\simeq\; C_{m+1}(F_i)\alpha_m/(\gamma_{m+1}(F) \cap D)\alpha_m \;.$$

Returning to the case where m = 2, we have a basis (1) for $C_3(F_i)\alpha_2$ so it is possible to use the above isomorphism to find a basis for $MG_3(A_i)$ by finding generators of $(\gamma_3(F) \cap D)\alpha_2$ expressed as products of the elements (1). To simplify the notation, put $u_i = x_i\alpha_2$ and $v_r = y_r\alpha_2$. Then the commutators,

$$\left.\begin{array}{ll} [u_i,u_j,v_r] & i > j \\ [v_r,u_i,u_j] & i \leq j \\ [v_r,u_i,v_s] & \end{array}\right\} \quad - \quad (2)$$

form a basis for $C_3(F_i)\alpha_2$, where $i,j=1, 2, \ldots, \ell$; $r,s=1, 2, \ldots, \ell'$.

Note that,

$$[v_r,u_i,v_s] = [u_i,v_s,v_r]^{-1}[v_s,v_r,u_i]^{-1} \qquad \text{by } (1.3.11)$$

$$= [[v_s,u_i]^{-1},v_r]^{-1}[[v_r,v_s]^{-1},u_i]^{-1}$$

$$= [v_s,u_i,v_r][v_r,v_s,u_i] \qquad \text{by } (1.3.10) \quad - \quad (a)$$

$$[v_r,u_i,u_j] = [v_i,v_r,u_j]^{-1} \qquad - \quad (b)$$

$$(\gamma_{m+1}(F)\cap D)\alpha_2 = \gamma_{m+1}(F\alpha_2)\cap D\alpha_2 \qquad - \quad (c)$$

and also, if $R_1 = T^{F_i}$ for some set T then,

$$([R_1,F_2]^F)\alpha_2 = \text{sgp}([y_r,z]^f \mid z \in T; f \in F; r = 1,2,\ldots,\ell')\alpha_2 \quad \text{by } (1.3.16)$$

$$= \text{sgp}([y_r,z],[y_r,z,f] \mid z \in T; f \in F; r = 1,2,\ldots,\ell')\alpha_2$$

$$= \text{sgp}([v_r,z\alpha_2],[v_r,z\alpha_2,h] \mid z \in T, h \in F\alpha_2, r = 1,2,\ldots,\ell') \qquad - \quad (d)$$

EXAMPLE 3.2.3 Suppose that $\ell = \ell' = 2$ and,

$$R_1 = \text{sgp}(x_1{}^3,x_2{}^3,[x_2,x_1,x_1],[x_2,x_1,x_2])^{F_1}$$

$$R_2 = \text{sgp}(y_1{}^3,y_1{}^3,[y_2,y_1,y_1],[y_2,y_1,y_2])^{F_2}.$$

Then $A_1 = F_1/R_1$, $A_2 = F_2/R_2$ are both isomorphic to the second nilpotent product of a pair of 3-cycles. The quotient $\gamma_1(A_1)/\gamma_2(A_1)$ is the direct product of a pair of 3-cycles and $\gamma_2(A_1)/\gamma_3(A_1)$ is a 3-cycle.

Now $([R_1,F_2]^F)\alpha_2$ is generated by the elements of the form,

$$[v_r,u_i{}^3],[v_r,u_i{}^3,h] \quad h \in F\alpha_2, \quad i,r = 1,2$$

We have by induction on k that,

$$[v_r,u_i{}^k] = [v_r,u_i]^k[v_r,u_i,u_i]^{k(k-1)/2} \quad \text{from (1.3.5) and (1.3.9)} .$$

Using this together with (1.3.9) implies that $([R_1,F_2]^F)\alpha_2$ is generated by the,

$$[v_r,u_i]^3[v_r,u_i,u_i]^6,[v_r,u_i,h]^3$$

and hence by,

$$[v_r,u_i]^3,[v_r,u_i,h]^3$$

It follows by symmetry that $([R_2,F_1]^F)\alpha_2$ is generated by the same elements and therefore so is $D\alpha_2$. But by Theorem 3.2.2, the $[u_i,v_r]$; $i,r = 1,2$ are a basis for $\gamma_2(F\alpha_2)/\gamma_3(F\alpha_2)$ and can be used in the obvious way to construct coset representatives for this group in $F\alpha_2$. Since $\gamma_3(F\alpha_2)$ is central in $F\alpha_2$ it is easy to verify that

$$\begin{aligned}\gamma_3(F\alpha_2)\cap D\alpha_2 &= \mathrm{sgp}([v_r,u_i,h]^3 \mid i,r = 1,2;\ h \in F\alpha_2) \\ &= \mathrm{sgp}([v_r,u_i,v_s]^3,[v_r,u_i,u_j]^3 \mid i,j,r,s = 1,2) .\end{aligned}$$

The identities (a) and (b) may be applied to show that this last subgroup is generated by the basis elements (2) each raised to the power 3. <u>$MG_2(A_i)$ is therefore the direct product of 16 3-cycles</u>.

EXAMPLE 3.2.4 Again suppose that $\ell = \bar{\ell} = 2$ and let R_2 be the same as in the previous example. This time put, $R_1 = \mathrm{sgp}(x_1{}^9,x_2{}^3,x_1{}^3[x_2,x_1])^{F_1}$, in which case $A_1 = F_1/R_1$ is the splitting extension of a 9-cycle by a 3-cycle with non trivial action and is nilpotent class 2. However

$$\gamma_k(A_1)/\gamma_{k+1}(A_1) \simeq \gamma_k(A_2)/\gamma_{k+2}(A_2), \quad k = 1,2 .$$

We have that $([R_1,F_2]^F)\alpha_2$ is generated by the elements,

$$\left.\begin{array}{l} [v_r,u_1^9],[v_r,u_2^3],[v_r,u_1^9,h],[v_r,u_2^3,h] \\ [v_r,u_1^3[u_2,u_1]],[v_r,u_1^3[u_2,u_1],h] \end{array}\right\} .$$

But

$$[v_r,u_1^3[u_2,u_1]] = [v_r,u_1^3][v_r,[u_2,u_1]] \quad \text{by (1.3.5) since}$$

$[v_r,[u_2,u_1]]$ is central .

and

$$[v_r,u_1^3[u_2,u_1],h] = [v_r,u_1^3,h] .$$

Therefore $([R_1,F_2]^F)\alpha_2$ is generated by,

$$\left.\begin{array}{l} [v_r,u_1]^9,[v_r,u_2]^3,[v_r,u_1,h]^9,[v_r,u_2,h]^3 \\ [v_r,u_1]^3[v_r,[u_2,u_1]] \end{array}\right\} .$$

Now $([R_2,F_1]^F)\alpha_2$ is generated by the same elements as before so that $D\alpha_2$ is generated by,

$$[v_r,u_i]^3,[v_r,u_i,h]^3,[u_2,u_1,v_r] .$$

The generators of $\gamma_{m+1}(F\alpha_2)\cap D\alpha_2$ are the same as in Example 3.2.3 except that the commutators $[y_2,y_1,v_r]$, $r = 1,2$, have been added. This means that $\underline{MG_2(A_i)}$ is now the direct product of 14 3-cycles.

EXAMPLE 3.2.5 Let A_1 be as in Example 3.2.4 and A_2 be isomorphic to A_1 with,

$$R_2 = sgp(y_1^9,y_2^3,y_1^3[y_2,y_1])^{F_2} .$$

We have that $D\alpha_2$ is generated by the elements,

$$\left.\begin{array}{l}
[v_r,u_1]^9, [v_r,u_2]^3, [u_i,v_1]^9, [u_i,v_2]^3 \quad - \text{(i)} \\[2ex]
\left.\begin{array}{l} [v_r,u_1,h]^9, [v_r,u_2,h]^3 \\ [u_i,v_1,h]^9, [u_i,v_2,h]^3 \end{array}\right\} \quad - \text{(ii)} \\[2ex]
\left.\begin{array}{l} [v_r,u_1]^3 [v_r,[u_2,u_1]] \\ [u_i,v_1]^3 [u_i,[v_2,v_1]] \end{array}\right\} \quad - \text{(iii)}
\end{array}\right\}$$

where $h \in F_2$; $i,r = 1,2$.

The presence of the generators (i) means that those of type (iii) may be replaced by,

$$\left.\begin{array}{l}
[v_1,u_1]^3 [v_1,[u_2,u_1]] \\
[u_1,v_1]^3 [u_1,[v_2,v_1]] \\
[u_1,u_2,v_1] \\
[v_2,v_1,u_1]
\end{array}\right\} \quad - \text{(iv)}$$

and hence by,

$$\left.\begin{array}{l}
[v_1,u_1]^3 [u_2,u_1,v_1]^{-1} \\
\left.\begin{array}{l} [v_2,v_1,u_1] \\ [u_1,u_2,v_1] \\ [v_2,v_1,u_1] \end{array}\right\} \quad - \text{(vi)}
\end{array}\right\} \quad - \text{(v)}$$

The intersection $\gamma_3(F\alpha_2) \cap D\alpha_2$ is therefore generated by the elements (ii) and (vi). All the elements (2) occur in $\gamma_3(F\alpha_2) \cap D\alpha_2$ to the power 3 except $[v_1,u_1,u_1]$ and $[v_1,u_1,v_1]$ which occur raised to the 9^{th} power. A careful enumeration of the possibilities shows that <u>$MG_2(A_i)$ is the direct product of a pair of 9-cycles by 11 3-cycles</u>.

3. The multiplicator of a finite nilpotent product of cyclic groups

Suppose that $A_1, A_2, \ldots, A_n$ are finite cyclic groups and that $G_m = \prod_{i=1}^{n}{}^{(m)} A_i$. Now since the $M(A_i)$ are trivial (see [14]), $M(G_m) \simeq MG_m(A_i)$ and, in calculating $M(G_m)$ it suffices to consider G_m as a finite p-group.

Let $N(m+1,i)$ be the number of symbols of weight $m+1$ in a basic sequence on $x_1, x_2, \ldots, x_i$. This is given by Witt's formula [17] page 330. In this section we prove the following two theorems.

THEOREM 3.3.1 Suppose A_i is a cyclic group of order p^{n_i} where $i = 1, 2, \ldots, n$ and $n_1 \geq n_2 \geq \ldots \geq n_n$. Then $M(\prod_{i=1}^{n}{}^{(m)} A_i)$ is an abelian group of type,

$$(\underbrace{p^{n_2}, p^{n_2}, \ldots, p^{n_2}}_{N(m+1,2)}, \underbrace{p^{n_3}, p^{n_3}, \ldots, p^{n_3}}_{N(m+1,3)\ -\ N(m+1,2)}, \ldots \quad \underbrace{\ldots, p^{n_n}, p^{n_n}, \ldots, p^{n_n}}_{N(m+1,n)\ -\ N(m+1,n-1)})$$

provided $1 \leq m \leq p-1$.

THEOREM 3.3.2 Suppose $A_1, A_2, \ldots, A_n$ satisfy the conditions of Theorem 3.3.1, then $M(\prod_{i=1}^{n}{}^{(p)} A_i)$ is the direct product of abelian groups of the type

$$(\underbrace{p^{n_i}, p^{n_i}, \ldots, p^{n_i}, \underbrace{p^{n_i-1}, p^{n_i-1}, \ldots, p^{n_i-1}}_{\chi(i)}}_{N(m+1,i)\ -\ N(m+1,i-1)})$$

for $i = 2, 3, \ldots, n$ where $\chi(i)$ is the number of elements in the set $\{j \mid 1 \leq j < i, n_i = n_j\}$.

An application of (1.3.20) shows that,

$$MG_m(A_i) \simeq C_{m+1}(F_i)\alpha_m / (\gamma_{m+1}(F) \cap R)\alpha_m$$

where (i) F_i/R_i is a presentation for A_i

(ii) $F = \prod_{i=1}^{n}{}^{*} F_i$

(iii) $R = \prod_{i=1}^{n} R_i^F$

(iv) $\alpha_m : F \to F/\gamma_{m+2}(F)$ is the canonical homomorphism.

Taking F_i to be the infinite cycle on f_i forces $\gamma_2(F) = C_2(F_i)$ by (2.1.5) since $\gamma_2(F_i) = E$. Therefore, by induction on m, $\gamma_{m+1}(F) = C_{m+1}(F_i)$ and hence from above,

(3.3.3) $M(G_m) \simeq \gamma_{m+1}(F)\alpha_m/(\gamma_{m+1}(F)\cap R)\alpha_m$.

We will work mainly with the group $X = F\alpha_p$. Note that $X = \prod_{i=1}^{n}{}^{(p+1)} X_i$ where $X_i = \mathrm{sgp}(x_i)$, $x_i = f_i\alpha_p$ (since $\gamma_{p+1}(F) = C_{p+1}(F_i)$).

Now the map $f\alpha_m \mapsto (f\alpha_p)\beta_m$, where β_m is the canonical homomorphism from X to $X/\gamma_{m+2}(X)$, $0 \leq m \leq p$, is an isomorphism of $\gamma_{m+1}(F)\alpha_m$ and $\gamma_{m+1}(X)\beta_m$ (by (1.3.20)) so that,

(3.3.4) $M(G_m) \simeq \gamma_{m+1}(X)\beta_m/(\gamma_{m+1}(X) \cap R\alpha_p)\beta_m$, $1 \leq m \leq p$.

Moreover if we take a sequence $c_1, c_2, c_3, \ldots$ of basic commutators in F on $f_1, f_2, \ldots, f_n$ and put $b_r = c_r\alpha_p$, $\mathrm{wt}b_r = \mathrm{wt}c_r$, then it follows from Theorem 3.2.2 that,

(3.3.5) $\gamma_{m+1}(X)\beta_m$ is a free abelian group with a basis consisting of the elements $b_r\beta_m$, $\mathrm{wt}b_r = m+1$, where $b_1, b_2, b_3, \ldots$ is a basic sequence on $x_1, x_2, \ldots, x_n$ whose bracket symbols are evaluated as commutators in X.

The b_r are distinct and non trivial for $wtb_r < p+2$ and $b_r = 1$ for $wtb_r \geq p+2$. From now on consider this sequence fixed and let $b_1, b_2, \ldots, b_\ell$ be the commutators of weight less than p+2.

Suppose A_i is of order p^{n_i}, then our choice of F_i makes $R_i = sgp(f^{p^{n_i}})$. It is convenient to impose the ordering $n_1 \geq n_2 \geq \ldots \geq n_n$.

We state the key result.

THEOREM 3.3.6 $R\alpha_p$ is generated by the following products of basic commutators.

$$\left.\begin{array}{ll}
x_i^{p^{n_i}} & \\
[x_i,x_j]^{p^{n_i}}[x_i,x_j,(p-1)x_i]^{\binom{p^{n_i}}{p}} & i > j,\ n_i \neq n_j \\
\left.\begin{array}{ll}
([x_i,x_j,(p-1)x_i][x_j,px_j]^{-1})^{\binom{p^{n_i}}{p}} & i > j,\ n_i = n_j \\
b_r^{p^{n_s}} & wtb_r > 2,\ b_r \in [X_s,X]
\end{array}\right\} - &
\end{array}\right\} \quad (*)$$

where i,j,r,s = 1, 2, ..., n and $\binom{p^{n_i}}{p} = p^{n_i}!/(p -p)!p!$. (NB: b_r may belong to $[X_s,X]$ for more than one s.)

A trivial consequence of this theorem is that $R\alpha_p\beta_m$, $1 \leq m < p-1$, is generated by the $(b_r\beta_m)^{p^{n_s}}$; s = 1, 2, ..., n (no restriction on wtb_r) where $b_r \in [X_s,X]$. This is equivalent via (3.1.11) to the finite case of a result by Struik on the multiplication of elements in an m^{th} nilpotent product of cycles (see [26] Theorem 3). The connection is quite clear since $G_m = \prod_{i=1}^{n}{}^{(m)} A_i$ is isomorphic to $X\beta_m/R\alpha_p\beta_m$. Both results amount to stating a set of defining relations for G_m on the generators $b_1\beta_m$, $b_2\beta_m$, $b_3\beta_m$, ... The full statement of Theorem 3.3.6 is presumably equivalent to [27] Theorem 5 although establishing the precise connection would not be so easy.

Rather than trying to deduce Theorem 3.3.6 from Struik's results, an independent proof of it is offered in section 4. The proof consists of a sequence of somewhat tedious inductive arguments but it avoids Struik's even lengthier numerical computations. It has other advantages in that it deals with the two cases of [26] and [27] simultaneously and ought to be easily generalised given more data about certain coefficients in Hall's collection process.

Now, Corollary 3.4.7 and the fact that $\gamma_{p+1}(X)$ is central in X implies that each element, x, of $R\alpha_p$ can be written in the form,

$$\left(\prod_{i>j}\left([x_i,x_j]^{p^{n_i}}[x_i,x_j,(p-1)]\right)^{m_{ij}}\right)x'$$

where some of the m_{ij} are possibly zero and x' is a product of the elements (*) (Theorem 3.3.6). But then,

$$\prod_{i>j}\left([x_i,x_j]\beta_3\right)^{m_{ij}} = 1$$

so that each $m_{ij} = 0$ by (3.3.5). It follows that $\gamma_2(X) \cap R\alpha_p$ is generated by the elements (*). Moreover the ordering $n_1 \geq n_2 \geq \ldots \geq n_n$ means that we may restrict the generators $b_r^{p^{n_s}}$, $\mathrm{wt}\,b_r > 2$, $b_r \in [X_s,X]$ to those of the form $b_r^{p^{n_t}}$ where $t = \max\{s \mid b_r \in [X_s,X]\}$ and still have a generating set for $\gamma_2(X) \cap R\alpha_p$. Put $\omega(r) = \max\{s \mid b_r \in [X_s,X]\}$. Then from (3.3.5), $(\gamma_{m+1}(X) \cap R\alpha_p)\beta_m$ $(= \gamma_{m+1}(X)\beta_m \cap R\alpha_p\beta_m)$ is generated by the elements $(b_r\beta_m)^{p^{n_t}}$ where $\mathrm{wt}\,b_r = m+1$, $t = \omega(r)$, provided $1 \leq m \leq p-1$. (A finite abelian group is said to be of type $(m_1, m_2, \ldots, m_k)$ where the m_i are prime powers if it has a basis consisting of elements of order $m_1, m_2, \ldots, m_k$ respectively.) From (3.3.4) and (3.3.5),

(3.3.7) $M(G_m)$, $1 \le m \le p-1$, is an abelian group of type

$$(\underbrace{p^{n_2}, p^{n_2}, \ldots, p^{n_2}}_{\mu(m+1,2)}, \underbrace{p^{n_3}, p^{n_3}, \ldots, p^{n_3}}_{\mu(m+1,3)}, \ldots \quad , \underbrace{p^{n_n}, p^{n_n}, \ldots, p^{n_n}}_{\mu(m+1,n)})$$

where $\mu(m+1,i)$ is the number of basic commutators, b_r, with $\mathrm{wt}\, b_r = m+1$ and $\omega(r) = i$.

In the exceptional case where $m = p$, $(\gamma_{p+1}(X) \cap R\alpha_p)\beta_p$ is generated by the $(b_r\beta_m)^{p^{n_t}}$ where $\mathrm{wt}\, b_r = p+1$, $t = \omega(r)$, together with the products,

$$([x_i, x_j, (p-1)x_i][x_i, px_j]^{-1})^{\binom{p^{n_i}}{p}} \qquad i > j,\ n_i = n_j\ .$$

Now $\binom{p^{n_i}}{p}$ is of the form qp^{n_i-1} where p does not divide q. The abelian group generated by a, b with defining relations $a^{p^{n_i}} = 1$, $b^{p^{n_i}} = 1$, $(ab^{-1})^{\binom{p^{n_i}}{p}} = 1$, must therefore have $a' = a^q$ and $b' = (ab)^q$ as a basis and be of type (p^{n_i}, p^{n_i-1}).

(3.3.8) $M(G_p)$ is the direct product of the abelian groups of type,

$$\underbrace{(p^{n_i}, p^{n_i}, \ldots, p^{n_i}, \underbrace{p^{n_i-1}, p^{n_i-1}, \ldots, p^{n_i-1}}_{\chi(i)}}_{\mu(p+1,i)})\ .$$

for $i = 2, 3, \ldots, n$ where $\chi(i)$ is the number of elements in the set $\{j \mid 1 \le j < i,\ n_i = n_j\}$.

It remains to be shown that $\mu(m+1,i) = N(m+1,i) - N(m+1,i-1)$. Consider the chain of subgroups,

$$X_1 = Y_1 < Y_2 < \ldots < Y_n = X$$

where $Y_i = \mathrm{sgp}(X_1, X_2, \ldots, X_i)$.

LEMMA 3.3.9 Each b_r, $\mathrm{wt}b_r > 1$ is contained in at least one of the groups $[X_s,Y_{s-1}]$; $s = 2, 3, \ldots, n$. If indeed $b_r \in [X_i,Y_{i-1}]$ then $i = \omega(r)$. (This means that each non trivial b_r, $\mathrm{wt}b_r > 1$, is in <u>exactly</u> one $[X_s,Y_{s-1}]$.)

Proof (i) We prove the first part by induction on $\mathrm{wt}b_r$. Each b_r of weight 2 is a commutator of the form $[x_s,x_t]$, $s > t$, and this is clearly in $[X_s,Y_{s-1}]$. Suppose $\mathrm{wt}b_r > 2$ then by Definition 3.2.1 and the inductive hypothesis, $b_r = [b_i,b_j]$ where $b_i \in [X_s,Y_{s-1}]$, $b_j \in [X_t,Y_{t-1}]$. It may be assumed without loss of generality that $s > t$, in which case,

$$\begin{aligned}
[[X_s,Y_{s-1}],[X_t,Y_{t-1}]] &\leq [[X_s,Y_{s-1}],Y_t] \\
&\leq [[X_s,Y_{s-1}],Y_{s-1}] \\
&\leq [X_s,Y_{s-1}] ,
\end{aligned}$$

and b_r is in $[X_s,Y_{s-1}]$.

(ii) Suppose that $b_r \in [X_i,Y_{i-1}]$. If $b_r \in [X_s,Y_{s-1}]$ for $s > i$, then $b_r \in (\bar{Y}_s)^X$ where $\bar{Y}_s = \mathrm{sgp}(X_s,X_{s+1}, \ldots, X_n)$. But Y_i is a subgroup of Y_{s-1} so that b_r is contained in both Y_{s-1} and $(\bar{Y}_{s-1})^X$. This is a contradiction since $X = Y_{s-1}(p+1)\bar{Y}_s$ by the associativity of nilpotent multiplications. Hence $\omega(r) = \max\{s | b_r \in [X_s,X]\} = i$. //

The rank of a free abelian group is the (unique) number of elements in a basis for that group. It follows from Lemma 3.3.9 that $\mu(m+1,i)$ is the number of b_r of weight $m+1$ in $[X_i, Y_{i-1}]$. This is equal to the rank of $(\gamma_{m+1}(X) \cap [X_i, Y_{i-1}])\beta_m$ by (3.3.5).

Now $X = Y_i(p+1)\bar{Y}_{i+1}$ and hence, from Lemma 3.1.6,

$$\gamma_{m+1}(X) = \gamma_{m+1}(Y_i)\gamma_{m+1}(\bar{Y}_{i+1})(\gamma_{m+1}(X) \cap [Y_i, \bar{Y}_{i+1}]^X) .$$

Thus,

$$\gamma_{m+1}(X) \cap Y_i = \gamma_{m+1}(Y_i) \qquad \text{by (2.1.3)}$$

and,

$$\gamma_{m+1}(X) \cap [X_i, Y_{i-1}] = \gamma_{m+1}(Y_i) \cap [X_i, Y_{i-1}] .$$

Proposition (1.3.20) may then be invoked to show that

$$(\gamma_{m+1}(X) \cap [X_i, Y_{i-1}])\beta_m \simeq (\gamma_{m+1}(Y_i) \cap [X_i, Y_{i-1}])\beta_{m,i}$$

where $\beta_{m,i}$ is the canonical homomorphism from Y_i to $Y_i/\gamma_{m+2}(Y_i)$.

Also $Y_i = Y_{i-1}(p+1)X_i$ and $\gamma_{m+1}(X_i) = E$ so that,

$$\gamma_{m+1}(Y_i) = \gamma_{m+1}(Y_{i-1})(\gamma_{m+1}(Y_i) \cap [X_i, Y_{i-1}]) .$$

Hence

$$\gamma_{m+1}(Y_i)\beta_{m,i} = \gamma_{m+1}(Y_{i-1})\beta_{m,i} \times (\gamma_{m+1}(Y_i) \cap [X_i, Y_{i-1}])\beta_{m,i} .$$

Denote the "rank of A" by rank A. Then we have from above that,

$$\begin{aligned}\mu(m+1,i) &= \operatorname{rank}(\gamma_{m+1}(X) \cap [X_i, Y_{i-1}])\beta_m \\ &= \operatorname{rank}(\gamma_{m+1}(Y_i) \cap [X_i, Y_{i-1}])\beta_{m,i} \\ &= \operatorname{rank} \gamma_{m+1}(Y_i)\beta_{m,i} - \operatorname{rank} \gamma_{m+1}(Y_{i-1})\beta_{m,i} \\ &= \operatorname{rank} \gamma_{m+1}(Y_i)\beta_{m,i} - \operatorname{rank} \gamma_{m+1}(Y_{i-1})\beta_{m,(i-1)}\end{aligned}$$

since $\gamma_{m+1}(Y_{i-1})\beta_{m,i} \simeq \gamma_{m+1}(Y_{i-1})\beta_{m,(i-1)}$ by (1.3.20).

The statement of (3.3.5) remains valid when X is replaced by Y_i and β_m is replaced by $\beta_{m,i}$ since this is only a change of parameter from n to i. The rank of $\gamma_{m+1}(Y_i)\beta_{m,i}$ is consequently the number N(m+1,i) of basic symbols of weight m+1 in a basic sequence on $x_1, x_2, \ldots, x_i$.

There is no need to change any proofs in order to generalise both Theorems 3.3.1 and 3.3.2 to include the case where $A_1, A_2, \ldots, A_k$, say, are infinite cycles. One simply adopts the usual convention that $n_1, n_2, \ldots, n_k$ are infinite. However a more careful analysis is required when G_m is allowed to be infinite and the torsion part of G_m is not a p-group.

4. Proof of Theorem 3.3.6

An immediate corollary to (3.3.5) is,

(3.4.1) (i) Each element $x \in X$ can be expanded uniquely as a product $x = b_1^{m_1} b_2^{m_2} \ldots b_\ell^{m_\ell}$ where the m_r are integers, possibly zero.

(ii) The product $b_1^{m_1} b_2^{m_2} \ldots b_\ell^{m_\ell}$ is in $\gamma_m(X)$ if and only if $m_r = 0$ for $\mathrm{wt}\, b_r < m$; $r = 1, 2, \ldots, \ell$.

In particular, when $x = (x_1x_2)^{p^{n_i}}$

(3.4.2) (Hall [11] Theorem 3.1) $(x_1x_2)^{p^{n_i}} = x_1^{p^{n_i}} x_2^{p^{n_i}} b_3^{\kappa_3} b_4^{\kappa_4} \ldots b_\ell^{\kappa_\ell}$

where

(i) p^{n_i} divides κ_r for $\mathrm{wt}\, b_r < p$

(ii) If $\kappa_r \neq 0$ then $b_r \in [X_1, X_2]$, $r = 3, 4, \ldots, \ell$.

The Greek letters $\kappa_1, \kappa_2, \ldots, \kappa_\ell$ are used to distinguish this expansion from the arbitrary form in (3.4.1). Naturally the κ_r depend on i but we will be working with a fixed i and an extra subscript would be unnecessarily cumbersome.

The normal subgroup $R = \prod_{i=1}^{n} R_i^F$ is generated by the elements $x_i^{p^{n_i}}$, $[x_i^{p^{n_i}}, f]$; $i = 1, 2, \ldots, n$; $f \in F$. Let,

$$P_{i,m} = \text{sgp}(b_r^{p^{n_i}} \mid b_r \in X_i^X, \text{wt}b_r \geq m) \quad \text{in } X .$$

We prove, in a series of simple steps, that

$$[x^{p^{n_i}}, x'] = [x,x']^{p^{n_i}} y[x,x',(p-1)x]^{\binom{p^{n_i}}{p}} \qquad \text{for } x \in \gamma_m(X) \cap X_i^X$$

where $y \in P_{i,(m+1)}$. This enables us to show that $P_{i,3}$; $i = 1,2, \ldots, n$ is a subgroup of $[R,F]\alpha_p$. It is then routine to establish Theorem 3.3.6.

Most arguments in the proof are by induction on $\text{wt}b_r$ or on some parameter relating to the expansion in (3.4.1)(i). We commence by making a refinement of (3.4.1)(i).

(3.4.3) Suppose $x = b_1^{m_1} b_2^{m_2} \ldots b_\ell^{m_\ell}$ belongs to X_i^X. Then if $m_r \neq 0$, b_r belongs to X_i^X for each $r = 1, 2, \ldots, \ell$.

Proof If $\bar{X}_i = \text{sgp}(X_j \mid j \neq 1)$ then $X = X_i(p+1)\bar{X}_i$ by the associativity of nilpotent multiplication. It can be shown by induction on $\text{wt}b_r$ that, for fixed i, each b_r is in X_i^X or $\bar{X}_i$. (The method of proof follows that of Lemma 3.3.9.) A given non trivial b_r cannot lie in both X_i^X and $\bar{X}_i$ by definition of a regular product.

Now $X = F/\gamma_{p+2}(F)$ is the relatively free nilpotent group of class p+1 on $x_1, x_2, \ldots, x_n$ so that the map $x_i \mapsto 1$, $x_j \mapsto x_j$, $j \neq 1$, induces an endomorphism τ of X ([21] 13.24) such that $(X_i^X)\tau = E$ and τ is the identity on $\bar{X}_i$.

Let $b_{r_1}, b_{r_2}, \ldots, b_{r_s}$ with $r_1 < r_2 < \ldots < r_s$ be the b_r in $\bar{X}_i$ and, if $b_r = b_{r_j}$, put $m_j' = m_r$. Clearly

$$b_{r_1}^{m_1'} b_{r_2}^{m_2'} \ldots b_{r_s}^{m_s'} = x\delta = 1$$

and each m_j' is zero by (3.4.1). //

(3.4.4) Suppose that δ is an endomorphism of X and that b_r is in X_j^X. Then,

(i) If $x_j\delta \in \gamma_m(X)$ and $\text{wt}b_r = m'$, then $b_r\delta \in \gamma_{m+m'-1}(X)$

(ii) If $x_j\delta \in X_i^X$ then $b_r\delta \in X_i^X$.

Proof When $m' = 1$, $b_r = x_j$ and the result is trivial. Suppose that it holds for all b_r with $\text{wt}b_r < m'$. If $\text{wt}b_r = m'$ then $b_r\delta = [b_s\delta, b_t\delta]$ where $\text{wt}b_s + \text{wt}b_t = m'$ by Definition 3.2.1. Now $[\gamma_u(X), \gamma_v(X)]$ is a subgroup of $\gamma_{u+v}(X)$ so that, by hypothesis, $b_r\delta$ is a subgroup of $\gamma_{m''}(X)$ where

$$\begin{aligned} m'' &= (m + \text{wt}b_s - 1) + (m + \text{wt}b_t - 1) \\ &= 2m + m' - 2 \\ &\leq m + m' - 1 \qquad \text{since } m \geq 1. \end{aligned}$$

Thus (i) is proved by induction on $\text{wt}b_r$. Part (ii) follows similarly since each b_r is in either X_j^X or $\bar{X}_j$ and $[[X_i,X],X]$ is a subgroup of $[X_i,X]$. //

The previous two propositions lead to a crucial lemma.

LEMMA 3.4.5 If $x \in \gamma_m(X) \cap X_i^X$ where $m \geq 3$, then $x^{p^{n_i}} \in P_{i,m}$.

Proof If $x = b_i^{m_1} b_2^{m_2} \dots b_\ell^{m_\ell}$, let $\sigma(x) = r$ where m_r is the first non zero integer of $m_1, m_2, \dots, m_\ell$. Note that b_r is in X_i^X by (3.4.3) and $\mathrm{wt}b_r = m$ by (3.4.1).

The proof is by induction on $\ell - \sigma(x)$. The lemma is trivially valid when $\sigma(x) = \ell$ because $x^{p^{n_i}} = b_\ell^{m_\ell p^{n_i}}$. Suppose it is valid for all $x' \in \gamma_m(X) \cap X_i^X$ such that $\sigma(x') < r$.

If $\sigma(x) = r$ then x may be written as $x = b_r^{m_r} y$, $m_r \neq 0$, where $y = b_{r+1}^{m_{r+1}} b_{r+2}^{m_{r+2}} \dots b_{r_\ell}^{m_\ell}$. The product y is also in $\gamma_m(X) \cap X_i^X$. Let δ be the endomorphism of X induced by the map

$$\left.\begin{array}{ll} x_1 \mapsto b_r^{m_r} & \\ x_2 \mapsto y & \\ x_j \mapsto x_j & j = 3, 4, \dots, n \end{array}\right\} .$$

Thus from (3.4.2),

$$x^{p^{n_i}} = b_r^{m_r p^{n_i}} y^{p^{n_i}} (b_3\delta)^{\kappa_3} (b_4\delta)^{\kappa_4} \dots (b_\ell\delta)^{\kappa_\ell} \qquad - \quad (1)$$

where p^{n_i} divides κ_s for $\mathrm{wt}b_s < p$, and $b_s \in [X_s, X]$ for $\kappa_s \neq 0$; $s = 3, 4, \dots, \ell$. Moreover, since $x_1\delta \in \gamma_3(X) \cap X_i^X$, (3.4.4) may be applied to show that $b_s\delta \in \gamma_{m'+m-1}(X) \cap X_i^X$, provided $\kappa_s \neq 0$, where $\mathrm{wt}b_s = m'$. This means that the $(b_s\delta)^{\kappa_s}$ with $\mathrm{wt}b_s \geq p$ can be eliminated from (1) because $m \geq 3$ and hence $b_s\delta$ is in $\gamma_{p+2}(X) = E$. That is,

$$x^{p^{n_i}} = b_r^{m_r p^{n_i}} y^{p^{n_i}} (b_3\delta)^{\kappa_3} (b_4\delta)^{\kappa_4} \dots (b_k\delta)^{\kappa_k} \qquad - \quad (2)$$

where p^{n_i} divides κ_s, $s = 3, 4, \dots, k$. But $b_r^{m_r p^{n_i}}$ is trivially in $P_{i,m}$ and $y^{p^{n_i}}$ is in $P_{i,m}$ by the inductive hypothesis since $\sigma(y) \leq r+1$. Finally, to deal with the $(b_s\delta)^{\kappa_s}$, we observe that any x' in $\gamma_{m+1}(X)$ necessarily has $\sigma(x') < r$ since $\mathrm{wt}b_r = m$. Therefore $(x')^{p^{n_i}} \in P_{i,m}$

for each $x' \in \gamma_{m+1}(X) \cap X_i^X$ by the inductive hypothesis. From above, $b_s\delta \in \gamma_{m+1}(X) \cap X_i^X$ when $\kappa_s \neq 0$ since $m' \geq 2$. Thus $(b_s\delta)^{\kappa_s} \in P_{i,m}$ for $\kappa_s \neq 0$. It follows from (2) that $x^{p^{n_i}} \in P_{i,m}$ and the result is proved by induction. //

We preface the next step with a brief description of how Hall's collection process is applied to $(x_1x_2)^m$, ($m = 2, 3, \ldots$) to expand it as a product $b_1^{m_1}b_2^{m_2} \ldots b_\ell^{m_\ell}$.

Any product yx $y,x \in X$ is equal to $xy[y,x]$. Therefore the product $y_1y_2 \ldots y_tx$ can be rewritten as $xy_1[y_1,x]y_2[y_2,x]\ldots y_t[y_t,x]$. The element x is said to be "moved to the left of the product" by introducing the commutators $[y_i,x]$ $i = 1, 2, \ldots, t$.

Consider the product,

$$(x_1x_2)^m = \underbrace{(x_1x_2)(x_1x_2) \ldots (x_1x_2)}_{m} .$$

Moving the second x_1 from the left to the extreme left of the product introduces the commutator $[x_2,x_1]$. Then taking the new product

$$x_1^2x_2[x_2,x_1]x_2\underbrace{(x_1x_2) \ldots (x_1x_2)}_{m-2}$$

and moving what is now the second x_1 from the left, in the same way, introduces commutators $[x_2,x_1]$ and $[x_2,x_1,x_1]$. The process can be repeated up to m-1 times until all the x_1's are "collected" to the left of the product and $(x_1x_2)^m = b_1^m y_{1,m}$ $(b_1 = x_1)$ where $y_{1,m}$ is a product of x_2's $(b_2 = x_2)$ and commutators of the form $[x_2,m'x_1]$, $1 \leq m' \leq m-1$. Definition 3.2.1 implies that each $[x_2,m'x_1]$ is a basic commutator. This is the first stage of the collection process.

Suppose that the r^{th} stage of the process has been reached. That is $(x_1x_2)^m$ has been collected into the form $b_1^{m_1}b_2^{m_2}\ldots b_r^{m_r}y_{r,m}$ where $y_{r,m}$ is a product of basic commutators, b_s, in some order such that $b_s > b_r$. We get to the $(r+1)^{th}$ stage as follows. If b_{r+1} does not appear in $y_{r,m}$ then the r^{th} and $(r+1)^{th}$ stages are the same. Otherwise collect all the b_{r+1}'s to the left of $y_{r,m}$. The commutators $[b_s,b_{r+1}]$, $b_s > b_{r+1}$, so introduced are all basic commutators necessarily greater than b_{r+1}. In both instances we have that

$$(x_1x_2)^m = b_1^{m_1}b_2^{m_2}\ldots b_r^{m_r}b_{r+1}^{m_{r+1}}y_{(r+1),m} \qquad (m_1 = m_2 = m)$$

where m_{r+1} is the number of b_{r+1}'s occurring in $y_{r,m}$, and $y_{(r+1),m}$ is some product of basic commutators greater than b_{r+1}. Thus the ℓ^{th} stage of the collection process can be reached inductively and $y_{\ell,m} = 1$ since $b_r = 2$ for $r > \ell$.

By construction, b_r, $n \leq r \leq \ell$, can be written uniquely as $[b_s,b_r]$ with $b_r > b_s > b_t$. Consequently b_r first appeared at the t^{th} stage of the collection process and no more b_r's could have been introduced at any later stage. <u>The power m_r must be equal to the number of b_r's in the product $y_{t,m}$.</u>

LEMMA 3.4.6 Suppose $x \in X_i^X \cap \gamma_m(X)$ and $x' \in \gamma_{m'}(X)$. Then,

$$[x^{p^{n_i}},x'] = [x,x']^{p^{n_i}}\, y[x,x',(p-1)x]^{\binom{p^{n_i}}{p}} \qquad \text{where } y \in P_{i,(m+m'+1)}\,.$$

Proof (i) (also in [26]) The identity $[x^m,x'] = x^{-m}(x[x,x'])^m$ can be established by induction on m. It is trivial for $m = 1$ and if it is true for some $m \geq 1$ then,

$$[x^{m+1},x'] = [x^m,x']^x[x,x'] \qquad \text{by (1.3.4)}$$

$$= x^{-1}(x^{-m}(x[x,x'])^m)x[x,x']$$

$$= x^{-(m+1)}(x[x,x'])^m(x[x,x'])$$

$$= x^{-(m+1)}(x[x,x'])^{m+1}.$$

(ii) Let δ be the endomorphism of X induced by the map,

$$\left.\begin{array}{ll} x_1 \mapsto x & \\ x_2 \mapsto [x,x'] & \\ x_j \mapsto x_j & j = 3, 4, \ldots, n \end{array}\right\}.$$

Then,

$$[x^{p^{n_i}},x'] = x^{-p^{n_i}}(x[x,x'])^{p^{n_i}} \qquad \text{by (i)}$$

$$= x_1^{-p^{n_i}}((x_1x_2)^{p^{n_i}})\delta$$

$$= [x,x']^{p^{n_i}}(b_3\delta)^{\kappa_3}(b_4\delta)^{\kappa_4}\ldots(b_\ell\delta)^{\kappa_\ell} \qquad - (1)$$

by (3.4.2) where p^{n_i} divides κ_s for $wtb_s < p$, and $b_s \in [X_1,X_2]$ for $\kappa_s \neq 0$; $s = 3, 4, \ldots, \ell$.

We simplify (1) using the same sort of argument as in the previous lemma. The situation is more delicate though, because neither $x_1\delta$ nor $x_2\delta$ is necessarily in $\gamma_3(X)$.

The commutator $[x,x']$ is in $\gamma_{m+m'}(X) \cap X_i^X$. Proposition (3.4.4) therefore implies that $b_s\delta \in \gamma_{m+m'+m''-1}(X) \cap X_i^X$ where $m'' = wtb_s$, and provided $\kappa_s \neq 0$. Note that $m+m' \geq 2$ since $m,m' \geq 1$. Now, $m'' \geq 2$ so that $b_s\delta \in \gamma_{m+m'+1}(X) \cap X_i^X$ for $\kappa_s \neq 0$. Thus $(b_s\delta)^{\kappa_s} \in P_{i,(m+m'+1)}$ for $wtb_s < p$, $\kappa_s \neq 0$ by Lemma 3.4.5. If $m'' = p+1$ then $b_s\delta \in \gamma_{p+2}(X) = E$ for $\kappa_s \neq 0$. We have that,

$$[x^{p^{n_i}},x'] = [x,x']^{p^{n_i}} y(b_k\delta)^{\kappa_k}(b_{k+1}\delta)^{\kappa_{k+1}} \ldots (b_{\ell'}\delta)^{\kappa_{\ell'}} \quad - (2)$$

from (1) where $y \in P_{i,m+1}$ and $s = k,k+1, \ldots, \ell'$ are the integers such that $wtb_s = p$.

(iii) We prove that the only non trivial $b_r\delta$ with $wtb_r = p$ and $\kappa_r \neq 0$ is $[x_2,(p-1)x_1]\delta$. Suppose that $wtb_r = p$ and $\kappa_r \neq 0$. Then from the discussion of the collection process, $b_r = [b_s,b_t]$ where $b_s > b_t$, $wtb_s + wtb_t = p$, $\kappa_s \neq 0$ and $\kappa_t \neq 0$.

If $b_t \neq x_1$ then $b_t = x_2$ or $b_t \in [X_1,X_2]$ by (3.4.2) since $\kappa_t \neq 0$. Moreover $b_s \in [X_1,X_2]$ since $b_s > b_t$. It follows by (3.4.4) that $b_s\delta \in \gamma_m(X)$ where $m = wtb_s+1$ and $b_t\delta \in \gamma_{m'}(X)$ where $m' = wtb_t+1$ (since $x_2\delta \in \gamma_2(X)$). Thus $b_r\delta = [b_s\delta,b_t\delta]$ belongs to $\gamma_{m''}(X)$ where

$$\begin{aligned} m'' &= m+m' \\ &= wtb_s + wtb_t+2 \\ &= p+2 \ . \end{aligned}$$

But $\gamma_{p+2}(X) = E$ so that the only non trivial $b_r\delta$ with $wtb_r = p$, $\kappa_r \neq 0$ is $b_r = [b_s,x_1]$. Part (ii)(a) of Definition 3.2.1 restricts such a b_r of weight p to be $[x_2,(p-1)x_1]$.

(iv) From (2) and (iii),

$$\begin{aligned} [x^{p^{n_i}},x'] &= [x,x']^{p^{n_i}} y([x_2,(p-1)x_1]\delta)^{\kappa} \\ &= [x,x']^{p^{n_i}} y[x,x',(p-1)x]^{\kappa} \qquad - (3) \end{aligned}$$

where κ is the power to which the basic commutator $[x_2,(p-1)x_1]$ occurs in the expansion $b_1^{p^{n_i}} b_2^{p^{n_i}} b_3^{\kappa_3} \ldots b_\ell^{\kappa_\ell}$ of $(x_1x_2)^{p^{n_i}}$.

It follows from the remarks preceding the statement of this lemma that $(x_1x_2)^m = x_1{}^m y_{1,m}$ at the first stage of the collection process where $y_{1,m}$ is is a product of basic commutators of the form $[x_1,m'x_1]$ with $1 \leq m' < m$. The power to which $[x_2,m'x_1]$ occurs in the final expansion of $(x_1x_2)^m$ is the number of times it occurs in the product $y_{1,m}$. We prove that this number is $\binom{m}{m'+1}$ by induction on m.

The proposition is vacuous for m = 1. Suppose it is true for some m > 1. Now, $(x_1x_2)^{m+1} = x_1{}^m y_{1,m}(x_1x_2)$ and this can be put into the first stage of the collection process $x_1{}^{m+1} x_2{}^{m+1} y_{1,(m+1)}$ by moving x_1 to the left of the product $y_{1,m}$. The commutator $[x_2,(m'+1)x_1]$ will therefore occur in $y_{1,(m+1)}$ the same number of times it occurs in $y_{1,m}$, plus the number of times $[x_2,m'x_1]$ (or x_2 when $m'+1 = 2$) occurs in $y_{1,m}$, $1 < m'+1 < m+1$. That is $\binom{m}{m'+1} + \binom{m}{m'}$ times by the inductive hypothesis. (There are $m = \binom{m}{1}$, x_2's in $y_{1,m}$.) But $\binom{m+1}{m'+1} = \binom{m}{m'+1} + \binom{m}{m'}$ so that the result holds by induction on m. In particular if $m = p^{n_i}$, $m' = p-1$, then $\kappa = \binom{p^{n_i}}{p}$. The final result holds by (3). //

COROLLARY 3.4.7 If $y \in P_{i,m}$, $m \geq 2$ and $x \in \gamma_{m'}(X)$ then $[y,x] \in P_{i,(m+m')}$.

Proof If $b_r \in \gamma_m(X) \cap X_i^X$ then $[b_r,x,(p-1)b_r] = 1$ since $m \geq 2$. Thus, by Lemma 3.4.6,

$$[b_r{}^{n'p^{n_i}},x] = [b_r{}^{n'},x]^{p^{n_i}} y' \quad \text{where } y' \in P_{i,(m+m'+1)} \text{ for any } n' > 0.$$

But $[b_r{}^{n'},x] \in \gamma_{m+m'}(X) \cap X_i^X$ so that $[b_r{}^{n'},x]^{p^{n_i}} \in P_{i,(m+m')}$ by Lemma 3.4.5. Consequently $[b_r{}^{n'p^{n_i}},x]$ is in $P_{i,(m+m')}$.

Now y must be of the form $y = b_{r_1}^{m_1} b_{r_2}^{m_2} \ldots b_{r_k}^{m_k}$ where $r_1 > r_2 > \ldots > r_k$, $b_{r_j} \in \gamma_m(X) \cap X_i^X$ and p^{n_i} divides m_j; $j = 1, 2, \ldots, k$. Hence

$$[y,x] = [b_{r_1}^{m_1},x]^{b_{r_2}^{m_2} b_{r_3}^{m_3} \ldots b_{r_k}^{m_k}} [b_{r_2}^{m_2},x]^{b_{r_3}^{m_3} \ldots b_{r_k}^{m_k}} \ldots [b_{r_k}^{m_k},x]$$

by (1.3.4) and each $[b_{r_j}^{m_j},x]$ is in $P_{i,(m+m')}$ from above. The result follows from this by another application of Lemma 3.4.5. //

THEOREM 3.4.8 $P_{i,3}$ is a subgroup of $R\alpha_p$ for each $i = 1, 2, \ldots, n$.

Proof (i) Put $S = R\alpha_p$. We prove first of all that if b_r is in X_i^X with $\mathrm{wt} b_r = m$, $m \geq 3$, then $b_r^{p^{n_i}}$ is in $SP_{i,(m+1)}$. When $m = 3$ there are three cases to consider.

(a) <u>$b_r = [b_s,x_i]$ where $\mathrm{wt} b_s = 2$</u>

In this instance $[b_s,x_i,(p-1)b_s] \in \gamma_{p+2}(X) = E$. Thus from Lemma 3.4.6,

$$[x_i^{p^{n_i}},b_s] = [x_i,b_s]^{p^{n_i}} y \quad \text{where } y \in P_{i,4} .$$

That is,

$$b_r^{p^{n_i}} = [x_i,b_s]^{-p^{n_i}}$$

$$= y[b_s,x_i^{p^{n_i}}] .$$

The commutator $[b_s,x_i^{p^{n_i}}]$ is in S since S is normal in X and $x_i^{p^{n_i}} \in S$.

(b) <u>$b_r = [x_i,x_j,x_k]$ $i > j$, $k \geq j$</u>

Now,

$$[x_i,x_j]^{p^{n_i}} = [x_i^{p^{n_i}},x_j][x_i,x_j,(p-1)x_j]^{-\binom{p^{n_i}}{p}} y^{-1} \quad \text{where } y \in P_{i,3} .$$

But $[x_i,x_j,(p-1)x_i]$ is in the central subgroup $\gamma_{p+1}(X)$ so that,

$$[[x_i,x_j]^{p^{n_i}},x_k] = [[x_i^{p^{n_i}},x_j]y^{-1},x_k]$$

$$= [[x_i^{p^{n_i}},x_j],x_k]^{y^{-1}}[y^{-1},x_k] \,.$$

Corollary 3.4.7 puts $[y^{-1},x_k]$ in $P_{i,4}$, and $[x_j^{p^{n_i}},x_j]$ is in S. Hence $[[x_i,x_j]^{p^{n_i}},x_k]$ is in $SP_{i,4}$. Finally $[[x_i,x_j]^{p^{n_i}},x_k] = [x_i,x_j,x_k]^{p^{n_i}}y'$ where $y' \in P_{i,4}$ since, $[[x_i,x_j],x_k,(p-1)[x_i,x_j]]$ is trivial. That is, $b_r^{p^{n_i}} = [[x_i,x_j]^{p^{n_i}},x_k](y')^{-1}$ is in $SP_{i,4}$.

(c) $\underline{b_r = [[x_i,x_j]^{-1},x_k] \quad i < j,\ k \leq i}$

This can be treated in the same way as (b).

Having established that $b_r \in SP_{i,(m+1)}$ for m = 3 we proceed by induction on m. Suppose $b_r \in X_i^X$, $m > 3$, then $b_r = [b_s,b_t]$ where at least one of b_s and b_t is in X_i^X, and $m = m'+m''$ where $m' = \mathrm{wt}b_s$, $m'' = \mathrm{wt}b_t$. If $b_s \in X_i^X$ then $b_s^{p^{n_i}}$ is of the form zy with $z \in S$, $y \in P_{i,(m'+1)}$ by the inductive hypothesis. Now $[b_s,b_t,(p-1)b_s] = 1$ since $[b_s,b_t] \in \gamma_3(X)$, and hence,

$$[b_s^{p^{n_i}},b_t] = [b_s,b_t]^{p^{n_i}}y' \quad \text{where } y' \in P_{i,(m+1)}$$

by Lemma 3.4.6. That is,

$$b_r^{p^{n_i}} = [zy,b_t]y'^{-1}$$

$$= [z,b_t]^y[y,b_t]y'^{-1} \,.$$

But $[z,b_t]^y$ is in S and $[y,b_t]$ is in $P_{i,(m'+1+m'')} = P_{i,(m+1)}$ by Corollary 3.4.7. Thus $b_r^{p^{n_i}}$ is in $P_{i,(m+1)}$. A similar proof works for $b_t \in X_i^X$.

(ii) It is immediate from (i) that if $P_{i,(m+1)} \leq S$ then $P_{i,m} \leq S$, $m \geq 3$. But $P_{i,(p+2)} = E$ is a subgroup of S so that $P_{i,m} \leq S$ for $3 \leq m \leq p+1$ by induction on ℓ-m. //

We restate and prove Theorem 3.3.6.

Theorem 3.3.6. $R\alpha_p$ is generated by the $P_{i,3}$, $i = 1, 2, \ldots, n$ together with all elements of the form,

$$\left.\begin{array}{ll} x_i^{p^{n_i}} & \\ [x_i,x_j]^{p^{n_i}} [x_i,x_j,(p-1)x_i]^{\binom{p^{n_i}}{p}} & n_i \neq n_j \\ ([x_i,x_j,(p-1)x_i][x_i,px_j]^{-1})^{\binom{p^{n_i}}{p}} & n_i = n_j \end{array}\right\}$$

where $i > j$; $i,j = 1, 2, \ldots, n$.

Proof Let $I = \{1, 2, \ldots, n\}$, $S = R\alpha_p$. By definition,

$$\begin{aligned} S &= \mathrm{sgp}(x_i^{p^{n_i}}, |i \in I)^X \\ &= \mathrm{sgp}(x_i^{p^{n_i}}, [x_i^{p^{n_i}}, x] | i \in I, x \in X) \\ &= \mathrm{sgp}(x_i^{p^{n_i}}, [x_i^{p^{n_i}}, x_j]^x | i,j \in I; x \in X) \quad \text{by (1.3.16) since the } x_j \text{ generate } X \\ &= \mathrm{sgp}(x_i^{p^{n_i}}, [x_i^{p^{n_i}}, x_j], [x_i^{p^{n_i}}, x_j, x] | i,j \in I, x \in X) . \end{aligned}$$

Now

$$\begin{aligned} [x_i^{p^{n_i}}, x_j, x] &= [[x_i,x_j]^{p^{n_i}} y, x] && \text{where } y \in P_{i,3} \text{ by Lemma 3.4.6 since } [x_i,x_j,(p-1)x_i] \text{ is central} \\ &= [y',x] && \text{where } y' \in P_{i,2} \text{ since either } [x_i,x_j] \text{ or } [x_j,x_i]^{-1} \text{ is a basic commutator.} \end{aligned}$$

Thus $[x_i^{p^{n_i}}, x_j, x]$ is in $P_{i,3}$ by Corollary 3.4.7. Conversely $P_{i,3} \leq S$ by Theorem 3.4.8.

Therefore,

$$S = \operatorname{sgp}(x_i^{p^{n_i}}, [x_i^{p^{n_i}}, x_j], y \mid y \in P_{i,3};\ i,j \in I)$$

$$= \operatorname{sgp}(x_i^{p^{n_i}}, z_{ij}, y \mid y \in P_{i,3};\ i,j \in I) \quad \text{again by Lemma 3.4.6 where}$$

$$z_{ij} = [x_i, x_j]^{p^{n_i}} [x_i, x_j, (p-1)x_i]^{\binom{p^{n_i}}{p}}$$

$$= \operatorname{sgp}(x_i^{p^{n_i}}, z_{ij}, z_{ji}, y \mid y \in P_{i,3};\ i > j;\ i,j \in I) \quad \text{since } z_{ii} = 1\,.$$

Put $\ell_i = \binom{p^{n_i}}{p}$. Then,

$$z_{ji} = [x_j, x_i]^{p^{n_j}} [x_j, x_i, (p-1)x_j]^{\ell_j}$$

$$= [x_i, x_j]^{-p^{n_j}} [x_i, x_j, (p-1)x_j]^{-\ell_j} \quad \text{by (1.3.10)}$$

$$= [x_i, x_j]^{-p^{n_j}} [x_i, px_j]^{-\ell_j}$$

and, for $i > j$, (i.e. $n_i \geq n_j$)

$$z_{ij} z_{ji}^{p^{n_i - n_j}} = [x_i, x_j, (p-1)x_i]^{\ell_i} [x_i, px_j]^{-\ell_j p^{n_i - n_j}}\,.$$

If $n_i = n_j$ then,

$$z_{ij} z_{ji}^{p^{n_i - n_j}} = ([x_i, x_j, (p-1)x_i][x_i, px_j]^{-1})^{\ell_j}\,.$$

Otherwise $z_{ij} z_{ji}^{p^{n_i - n_j}} \in P_{j,(p+1)}$ since p^{n_j} divides both $\ell_i = \binom{p^{n_i}}{p}$ and $\ell_j p^{n_i - n_j}$ ($p^{n_j - 1}$ divides ℓ_j). But from above,

$$S = \operatorname{sgp}(x_i^{p^{n_i}}, z_{ij}, z_{ij} z_{ji}^{p^{n_i - n_j}}, y \mid y \in P_{i,3};\ i > j;\ i,j \in I)$$

which gives the required result. //

CHAPTER 4

THE MULTIPLICATOR OF A SPLITTING EXTENSION

1. General results

DEFINITION 4.1.1 G is a splitting extension (semi direct product) of the group A by the group B if,

(i) G is generated by A and B (or isomorphic copies of them)

(ii) A is normal in G

(iii) $A \cap B = E$.

Since A is normal in G, the maps $b\theta : a \to a^b$, $a \in A$, for each $b \in B$, are automorphisms and they induce a homomorphism $\theta : B \to \mathrm{Aut}A$ called the action of B on A. G is determined up to isomorphism by θ and is therefore called the splitting extension of A by B under θ [10] 6.5. Note that every element of G is uniquely of the form ab, $a \in A$, $b \in B$.

The structure of $M(G)$ cannot be broken up as completely as for the case where G is a regular product of A and B. The main theorem of this chapter states that $M(B)$ is contained in $M(G)$ as a direct factor. Not much can be said in general about the complementary factor of $M(B)$. We spend the last part of this chapter and the subsequent chapter examining its structure with certain restrictions on A, B and θ.

Tahara [28] working independently, has proved a generalisation of the main theorem by showing that $H^2(B,C)$ is a direct factor of $H^2(G,C)$ for any trivial G-module C. ($H^2(G,C) \simeq M(G)$ when C is the rationals modulo Z.) The complementary factor of $H^2(B,C)$ is the kernel of a certain restriction homomorphism. His method involves a careful examination of the 2-cocycles from G to C.

Our approach to calculating M(G) is strictly analogous to that of Chapter 2 for regular products. Lemmas 4.1.4 and 4.1.6 correspond in turn to Lemmas 2.2.1 and 2.2.3.

If K is a regular product of its subgroups A and B then, according to Definitions 2.1.1 and 4.1.1, it is also a splitting extension of both A[A,B] by B and B[A,B] by A. (The subgroup A[A,B] is a splitting extension of [A,B] by A and similarly for B[A,B].) Unfortunately the converse is not true. For example, the symmetric group on three symbols is a splitting extension of a pair of cycles but a quick check shows that it cannot be expressed as a regular product in any way.

There are several reasons for not dealing with the multiplicator of a regular product as a special case of the theory derived below for splitting extensions. Firstly it is more difficult and less illuminating to try to prove that the complement of M(B) in M(K) is $M(A) \times MK(A,B)$ than to start with a different presentation for K as in Chapter 2. Secondly one would only obtain the multiplicator of a regular product of a <u>pair</u> of groups. Theorem 2.2.6 (for K a regular product of the A_λ, $\lambda \in I$) would then have to be established by transfinite induction on the index set I and it would be extremely hard to work with $MK(A_\lambda)$ in this context.

We start with a couple of elementary results on splitting extensions.

(4.1.2) Suppose G is the splitting extension of A by B under θ and $\bar{G}$ is the splitting extension of $\bar{A}$ by $\bar{B}$ under $\bar{\theta}$. If $\alpha : A \to \bar{A}$ and $\beta : B \to \bar{B}$ are epimorphisms such that,

$$(a(b\theta))\alpha = (a\alpha)(b\beta\bar{\theta}) \quad \text{for all } a \in A,\ b \in B\ ,$$

then the map,

$$\left.\begin{aligned} \tau : G &\to \bar{G} \\ ab &\mapsto (a\alpha)(b\beta) \end{aligned}\right\}$$

is an epimorphism of G extending α and β.

Proof It is routine to prove $((a_1b_1)(a_2b_2))\tau = (a_1b_1)\tau(a_2b_2)\tau$ using the fact that $b_1a_2 = (a_2(b_1^{-1}\theta))b_1$. //

(4.1.3) Suppose G is the splitting extension of A by B under θ. If N is a subgroup of A which is normal in G, then G/N is the splitting extension of A/N by B under the action,

$$(aN)^b = (a^b)N\ , \quad a \in A,\ b \in B\ .$$

Proof If α is the canonical homomorphism from A onto N then $N(b\theta)\alpha = E$, $b \in B$, since $N(b\theta) = N$. The homomorphism $b\theta\alpha : A \to A/N$ therefore induces an automorphism $b\phi : A/N \to A/N$. It is easily checked that the map,

$$\left.\begin{aligned} \phi : B &\to \mathrm{Aut}(A/N) \\ b &\mapsto b\phi \end{aligned}\right\}$$

is a homomorphism. Let $\bar{G}$ be the splitting extension of A/N by B under ϕ. Then, by construction,

$$\begin{aligned} a(b\theta\phi) &= (aN)b\phi \\ &= a\alpha(b\phi) \end{aligned}$$

and (4.1.2) may be invoked to show that the map $\tau : ab \mapsto (a\alpha)b$ from G onto $\bar{G}$ is a homomorphism. The kernel of τ is N. //

Proposition (4.1.3) can be used to find a presentation for G.

Lemma 4.1.4 Suppose G is a splitting extension of A by B under θ and F_1/R_1 and F_2/R_2 are presentations for A and B respectively, then F/R is a presentation for G where

(i) $F = F_1 * F_2$

(ii) $R = R_1^F R_2^F S^F$

(iii) $S = \mathrm{sgp}(f_1^{-1}\bar{f}_1[f_2,f_1] \mid f_1,\bar{f}_1 \in F_1; f_2 \in F_2; \bar{f}_1\nu_1 = (f_1\nu_1)(f_2\nu_2\theta))$

(iv) ν_1 and ν_2 are the epimorphisms from F_1, F_2 onto A, B.

Proof Let ψ be the natural homomorphism from F to $A * B$ induced by ν_1 and ν_2, and ϕ be the natural homomorphism from $A * B$ onto G induced by the identity maps on A and B. The kernel of ψ equals $R_1^F R_2^F$ by Lemma 2.2.1(i) so that $R = \ker\psi\phi$ if $(S^F)\psi = \ker\phi$.

Now, put $T = (S^F)\psi$. If $a = f_1\nu_1$, $b = f_2\nu_2$ and $\bar{f}_1 \in F_1$ such that $\bar{f}_1\nu_1 = a(b\theta)$, then,

$$\begin{aligned}(f_1^{-1}\bar{f}_1[f_2,f_1])\psi\phi &= (a^{-1}(a(b\theta))[b,a])\phi \\ &= a^{-1}a^b[b,a] \qquad \text{in } G \\ &= [a,b][b,a] \\ &= 1 \quad .\end{aligned}$$

This implies that $T\phi = E$ and hence T is a subgroup of $\ker\phi$.

To prove the converse we consider $A * B/T$. Note that T is a subgroup of $A[A,B]$ since $S\psi$ is clearly a subgroup of $A[A,B]$ and the latter is normal in $A * B$. In particular AT/T is a subgroup of $A[A,B]/T$. But, given $a \in A$, $b \in B$,

$$\begin{aligned}[b,a]T &= ([b,a](a^{-1}(a(b\theta))[b,a])^{-1})T \\ &= ((a(b\theta))^{-1}a)T\end{aligned}$$

so that $[A,B]T/T$ is a subgroup of AT/T. It follows that $AT/T = A[A,B]/T$.

The free product $A * B$ is a splitting extension of $A[A,B]$ by B which implies by (4.1.3) that $A * B/T$ is a splitting extension of $A[A,B]/T$ by B. The action of B on $A[A,B]/T$ (= AT/T) is given by,

$$\begin{aligned}(aT)^b &= (a^b)T \\ &= (a[b,a]^{-1})T \\ &= (aT)((a^{-1}(a(b\theta)))T) \\ &= (a(b\theta))T \quad .\end{aligned}$$

An application of (4.1.2) to $A * B/T$, with β equal to the identity on B and α equal to the isomorphism $aT \mapsto a$, shows that $A * B/T$ is isomorphic to G under the map $\tau : (aT)b \mapsto ab$. The diagram,

$$\begin{tikzcd} A * B \arrow[r, "\sigma"] \arrow[dr, "\phi"'] & A * B/T \arrow[d, "\tau"] \\ & G \end{tikzcd}$$

commutes by (4.1.3) where σ is the canonical homomorphism. Therefore,

$$\ker\phi = \ker\sigma\tau$$

$$= T$$

$$= (S^F)\psi \quad \text{as required .} \qquad //$$

The next two propositions put $R = R_1{}^F R_2{}^F S^F$ into a suitable form for calculating $R \cap F'/[R,F]$.

(4.1.5) R_1 and $[R_2,F_1]$ are subgroups of S.

Proof (i) If $r_1 \in R_1$ then $r_1\nu_1 = 1$ and hence $(r_1\nu_1)(f_2\nu_2\theta) = 1$ for each $f_2 \in F_2$. It follows that $r_1 = 1^{-1}r_1[f_2,1]$ is an element of S.

(ii) If $r_2 \in R_2$ then $r_2\nu_2 = 1$ is the identity automorphism. Therefore $[r_2,f_1] = f_1{}^{-1}f_1[r_2,f_1]$ is in S. //

Lemma 4.1.6 (i) $R = R_2S^F$

(ii) $R \cap F' = (R_2 \cap F_2{}')(S^F \cap F')$

(iii) $[R,F] = [R_2,F_2][R_2,F_1][S,F]$.

Proof (i)
$$\begin{aligned} R &= R_2{}^F R_1{}^F S^F \\ &= R_2{}^F S^F && \text{by (4.1.5)} \\ &= R_2[R_2,F_1]^F S^F && \text{by Lemma 2.2.2} \\ &= R_2S^F \quad . \end{aligned}$$

(ii)
$$\begin{aligned} R \cap F' &= R_2S^F \cap F_2{}'F_1{}'[F_1,F_2] && \text{by (i) and (2.1.5)} \\ &= (R_2 \cap F_2{}')(S^F \cap F_1{}'[F_1,F_2]) && \text{since } F \text{ is a splitting extension of } F_1[F_1,F_2] \text{ by } F_2 \text{ and } S^F \le F_1[F_1,F_2] \\ &= (R_2 \cap F_2{}')(S^F \cap F') \quad . \end{aligned}$$

(iii) $[R,F] = [R_2S^F,F]$

$= [R_2,F][S^F,F]$ by (1.3.17)

$= [R_2,F_2][R_2,F_1]^F[S,F]$ by (2.2.3) and (1.3.15)

$= [R_2,F_2][R_2,F_1][S,F]$ since $[[R_2,F_1],F] \leq [S,F]$ by (4.1.5). //

The main result can now be proved.

THEOREM 4.1.7 Suppose G is the splitting extension of A by B under θ then M(G) is isomorphic to,

$$M(B) \times S^F \cap F'/[R_2,F_2][S,F]$$

where F, F_2, R_2 and S are as defined in (4.1.4).

Proof (i) Let ξ and η be the canonical homomorphisms from F onto $F/[R_2,F_2]^F$ and $F\xi$ onto $F\xi/([R_2,F_1][S,F])\xi$ respectively. Then from Lemma 4.1.6,

$$R\cap F'/[R,F] = (R\cap F')\xi\eta$$

$$= (R_2 \cap F_2')\xi\eta(S^F \cap F')\xi\eta \ .$$

Since $R\cap F'/[R,F]$ is abelian, we only need to show that $(R_2 \cap F_2')\xi\eta$ and $(S^F \cap F')\xi\eta$ intersect trivially for $R\cap F'/[R,F]$ to be their direct product.

Now $F\xi = F_1 * (F_2/[R_2,F_2])$ by Lemma 2.2.1(i). Also the kernel, $([R_2,F_1][S,F])\xi$, of η is normal in $F\xi$ and contained in $F_1\xi[F_2\xi,F_2\xi]$. This implies, from (4.1.3), that $F\xi\eta$ is a splitting extension of $F_1\xi\eta[F_1\xi\eta,F_2\xi\eta]$ by $F_2\xi\eta$. In particular,

$$(R_2 \cap F_2')\xi\eta \cap (S^F \cap F')\xi\eta \le F_2\xi\eta \cap F_1\xi\eta[F_1\xi\eta, F_2\xi\eta] = E .$$

(ii) The kernel of $\xi\eta$ is $[R_2,F_2][R_2,F_1][S,F]$. Now $[R_2,F_2]$ and $R_2 \cap F_2'$ are contained in F_2 and $[R_2,F_1][S,F]$ is contained in $F_1[F_1,F_2]$. Thus $R_2 \cap F_2'$ intersects $\ker\xi\eta$ in $[R_2,F_2]$ and $(R_2 \cap F_2')\xi\eta \simeq R_2 \cap F_2'/[R_2,F_2] = M(B)$. Similarly $S^F \cap F'$ intersects $\ker\xi\eta$ in $[R_2,F_1][S,F]$ and $(S^F \cap F')\xi\eta \simeq S^F \cap F'/[R_2,F_1][S,F]$. //

The problem of finding $M(G)$ now boils down to finding the structure of the abelian group $S^F \cap F'/[S,F]$. Denote this group by $SG(A,B)$.

THEOREM 4.1.8 $SG(A,B)$ is isomorphic to $T^K \cap K'/[T,K]$ where $K = F_1 * B$ and,

$$T = sgp(f_1^{-1}\bar{f}_1[b,f_1] \mid f_1, \bar{f}_1 \in F_1; b \in B; \bar{f}_1\nu_1 = f_1\nu_1(b\theta)) .$$

Proof Let δ be the natural homomorphism from F onto K induced by ν_2 and the identity on F_1. If $b = f_2\nu_2$ and $\bar{f}_1\nu_1 = f_1\nu_1(b\theta)$ then,

$$\begin{aligned}(f_1^{-1}\bar{f}_1[f_2,f_1])\delta &= (f_1\delta)^{-1}(\bar{f}_1\delta)[f_2\delta,f_1\delta] \\ &= f_1^{-1}\bar{f}_1[b,f_1] .\end{aligned}$$

Thus $S\delta = T$ and hence $[S,F]\delta = [T,K]$. Suppose we could show that $(S^F \cap F') = T^K \cap K'$ then,

$$\begin{aligned}T^K \cap K'/[T,K] &= (S^F \cap F')\delta/[S,F]\delta \\ &\simeq S^F \cap F'/[R_2,F_1][S,F] \quad \text{by (1.3.20) since,}\end{aligned}$$

$$
\begin{aligned}
((S^F \cap F') \cap \ker\delta)[S,F] &= (S^F \cap F' \cap R_2{}^F)[S,F] \\
&= (S^F \cap F' \cap R_2[R_2,F_1]^F)[S,F] && \text{by Lemma 2.2.2} \\
&= (S^F \cap F' \cap [R_2,F_1]^F)[S,F] && \text{since } S^F \leq F_1[F_1,F_2] \\
&= [R_2,F_1]^F[S,F] && \text{by (4.1.5)} \\
&= [R_2,F_1][S,F] && \text{since } [[R_2,F_1],F] \leq [S,F] .
\end{aligned}
$$

That is, $T^K \cap K'/[T,K] \simeq SG(A,B)$.

To complete the proof it has to be shown that $(S^F \cap F')\delta$ is indeed equal to $T^K \cap K'$. Clearly $(S^F \cap F')\delta$ is a subgroup of $T^K \cap K'$ $(= S^F\delta \cap F'\delta)$. Conversely if $u \in T^K \cap K'$ then there exists $x \in S^F$, $y \in F'$ such that $x\delta = u = y\delta$. Now $yx^{-1} \in \ker\delta = R_2{}^F$. Therefore $y \in R_2{}^F S^F = R$. But $y \in F'$ so that $y \in R \cap F' = (R_2 \cap F_2{}')(S^F \cap F')$. This implies that $u = y\delta \in (S^F \cap F')$ since $(R_2 \cap F_2{}')\delta = E$. That is $T^K \cap K'$ is contained in $(S^F \cap F')\delta$. //

Theorem 4.1.8 provides a considerable simplification of $SG(A,B)$ and relates its structure more closely to that of B. But how does $SG(A,B)$ depend on the structure of A? If we identify $SG(A,B)$ with $T^K \cap K'/[T,K]$ then it contains $(R_1 \cap F_1{}')[T,K]/[T,K]$ since R_1 is a subgroup of T by (4.1.5). Moreover $[R_1,F_1]$ is in $[T,K]$ so that <u>$(R_1 \cap F_1{}')[T,K]/[T,K]$ is a homomorphic image of $M(A)$.</u>

(4.1.9) The quotient of $SG(A,B)$ by $(R_1 \cap F_1{}')[T,K]/[T,K]$ is isomorphic to $U^C \cap C'/[U,C]$ where $C = A * B$ and, $U = sgp(a^{-1}(a(b\theta))[b,a] \mid a \in A, b \in B)$.

Proof Let ξ be the natural homomorphism from K to C induced by ν_1 and the identity on B. The image of T under ξ is U. This time $\ker\xi = R_1^F$ is a subgroup of T^K so that $(T^K \cap K')\xi = U^C \cap C'$ immediately. Proposition (1.3.20) implies that,

$$U^C \cap C'/[U,C] \;=\; (T^K \cap K')\xi/[T,K]\xi$$

$$\simeq\; T^K \cap K'/(R_1 \cap F_1')[T,K]$$

since

$$\begin{aligned}(T^K \cap K' \cap \ker\xi)[T,K] &= (T^K \cap K' \cap R_1^K)[T,K] \\ &= (K' \cap R_1^K)[T,K] \qquad \text{since } R_1 \leq T \\ &= (K' \cap R_1[R_1,B]^K)[T,K] \\ &= (R_1 \cap F_1')[R_1,B]^K[T,K] \\ &= (R_1 \cap F_1')[T,K] \quad .\end{aligned}$$

Thus,

$$U^C \cap C'/[U,C] \;\simeq\; \frac{T^K \cap K'/[T,K]}{(R_1 \cap F_1')[T,K]/[T,K]} \;. \qquad //$$

If $M(A) = E$ then $(R_1 \cap F_1')[T,K]/[T,K] = E$.

COROLLARY 4.1.10 If $M(A) = E$ then $SG(A,B) \simeq U^C \cap C'/[U,C]$.

Not much more can be said in general about SG(A,B). Obviously there is insufficient information to construct a representing pair for finite G in the style of Theorem 2.3.1. However we can establish the existence of a representing group which is a splitting extension of a group L_1 by a representing group for B. L_1 is not a representing group for A but it contains SG(A,B) as a central subgroup and their quotient is isomorphic to A.

Theorem 23.5 in [14] Kapitel V states that there is a subgroup $D(R,F)$ of $F/[R,F]$ such that,

$$\text{(i)}\quad R/[R,F] = D(R,F) \times R\cap F'/[R,F] \qquad - (1)$$

$$\text{(ii)}\quad \left(\frac{F/[R,F]}{D(R,F)}, \frac{(R\cap F'/[R,F])D(R,F)}{D(R,F)} \right) \text{ is a representing pair for } G$$

when G is finite.

Let ρ be the canonical homomorphism from F onto $F/[R,F]$. Starting with the fact that $R = R_2S^F$, and copying the argument of Theorem 4.1.7 it can be shown that,

$$R\rho = R_2\rho \times (S^F)\rho . \qquad - (2)$$

But, replacing F and R by F_2 and R_2 in (1), we have

$$R_2/[R_2,F_2] = D(R_2,F_2) \times R_2 \cap F_2'/[R_2,F_2]$$

and, since $F_2\rho \simeq F_2/[R_2,F_2]$,

$$R_2\rho = \bar{D}(R_2,F_2) \times (R_2 \cap F_2')\rho \qquad - (3)$$

where $\bar{D}(R_2,F_2)$ is the image of $D(R_2,F_2)$. From Theorem 4.1.7,

$$(R\cap F')\rho = (R_2 \cap F_2')\rho \times (S^F \cap F')\rho . \qquad - (4)$$

Substituting (4) in (1) and (3) in (2), we have

$$D(R,F) \times (R_2 \cap F_2')\rho \times (S^F \cap F')\rho = \bar{D}(R_2,F_2) \times (R_2 \cap F_2')\rho \times (S^F)\rho . \qquad - (5)$$

Therefore, from (4) and (5),

$$(S^F)\rho = D(S) \times (S^F \cap F')\rho \qquad - (6)$$

where $D(S) = (S^F)\rho \cap D(R,F)$, and hence,

$$D(R,F) = \bar{D}(R_2,F_2) \times D(S) \qquad - (7)$$

since $(S^F \cap F')\rho$ and $(R_2 \cap F_2')\rho$ intersect $D(R,F)$ trivially by (1).

$F\rho$ is a splitting extension of $(F_1[F_1,F_2])\rho$ by $F_2\rho$ from Theorem 4.1.7. Put $L_1 = (F_1[F_1,F_2])\rho/D(S)$. Then by (6) and an application of (4.1.3), $F\rho$ is a splitting extension of L_1 by $F_2\rho/\bar{D}(R_2,F_2)$ which is what we were aiming at since $F\rho/D(R,F)$ is a representing group for G by (ii) and $F\rho/D(R,F) \simeq \dfrac{F_2/[R_2,F_2]}{D(R_2,F_2)}$ is a representing group for B.

Note that,

$$SG(A,B) = (S^F \cap F')\rho \quad \text{from Theorem 4.1.7}$$

$$\simeq (S^F)\rho/D(S) \quad \text{by (6)} \quad .$$

If we identify $SG(A,B)$ with $(S^F)\rho/D(S)$ then,

$$L_1/SG(A,B) = \frac{(F_1[F_1,F_2])\rho/D(S)}{(S^F)\rho/D(S)}$$

$$\simeq (F_1[F_1,F_2])\rho/(S^F)\rho$$

$$\simeq F_1[F_1,F_2]R/R \qquad \text{by (1.3.20) since,}$$

$$(F_1[F_1,F_2] \cap \ker\rho)S^F = [R_2,F_1]^F[S,F]S^F$$

$$= F_1[F_1,F_2] \cap R \ .$$

Thus $L_1/SG(A,B) \simeq A$ by Lemma 4.1.4. It is clear that $SG(A,B)$ is central in L_1 since $[S,F]$ is contained in $\ker\rho$.

There is a result on $SG(A,B)$ reminiscent of Theorem 2.3.1 which will be used in Chapter 5.

(4.1.11) When A is finite there exists a representing group L for A such that $SG(A,B)$ is isomorphic to $W^J \cap J'/[W,J]$ where

(i) $J = L * B$

(ii) $W = \mathrm{sgp}(\ell_1^{-1}\ell_2[b,\ell_1] \mid \ell_1,\ell_2 \in L; b \in B; \ell_2\omega = \ell_1\omega(b\theta))$

(iii) ω is the epimorphism of L onto A.

Proof Let χ be the natural homomorphism from $K = F_1 * B$ onto $(F_1/[R_1,F_1]) * B$. Then $L = F_1\chi/D$ is a representing group for A where,

$$R_1\chi = D \times (R_1 \cap F_1')\chi .$$

Let μ be the natural homomorphism from $K\chi$ onto $J = L * B$. Note that $T\chi\mu = W$. We prove $W^J \cap J'/[W,J] \simeq T^K \cap K'/[T,K]$.

Now,

$$(T^K \cap K')\chi/[T,K]\chi \simeq (T\chi)^{K\chi} \cap (K\chi)'/[T\chi,K\chi] \quad \text{since } R_1 \leq T \text{ and}$$
$$\ker\chi = [R_1,F_1]^K \leq T^K \qquad - \text{ (i)}$$

and

$$(W^J \cap J')/[W,J] = (T\chi)^{K\chi} \cap (K\chi)'/[T\chi,K\chi] \quad \text{since } \ker\mu = D^{K\chi}$$
$$\leq R_1\chi^{K\chi}$$
$$\leq T\chi^{K\chi} . \ - \text{ (ii)}$$

But

$$\begin{aligned} \ker\mu \cap (K\chi)' &= D^{K\chi} \cap (K\chi)' \\ &= D[D,K\chi] \cap (K\chi)' \\ &= (D \cap (F_1\chi)'B'[F_1\chi,B])[D,K\chi] \\ &= [D,K\chi] \qquad \text{since } D \cap (F_1\chi)' = E . \end{aligned}$$

Thus, by (1.3.20),

$$((T\chi)^{K\chi} \cap (K\chi)')\mu/[T\chi,K\chi]\mu \simeq (T\chi)^{K\chi} \cap (K\chi)'/[T\chi,K\chi] . \ - \text{ (iii)}$$

The result follows from (i), (ii), (iii) and Theorem 4.1.8. //

2. The multiplicator of a verbal wreath product

The wreath product of a pair of groups is a standard construction useful in formulating counter examples. Verbal wreath products are a natural generalisation in the context of variety theory. Both are defined in terms of a splitting extension.

Given arbitrary groups A and B, let A_b be an isomorphic copy of A for each $b \in B$. Denote the isomorphism $A \to A_b$ by α_b and let $C = \prod^*_{b\in B} A_b$. Suppose V is a given set of words in F_∞. Put $G_V = \prod^V_{b\in B} A_b$. The isomorphisms $\alpha_b^{-1}\alpha_{bb'} : A_b \to A_{bb'}$, $b \in B$, induce an automorphism $b'\theta_*$ of C. Lemma 3.1.4 (i) shows that $C_V = V(C) \cap [A_b{}^C]$ is left invariant by $b'\theta_*$ so that $b'\theta_*$ induces an automorphism $b\theta_V$ of $G_V = C/C_V$. It is easily checked that $\theta_V : B \to \mathrm{Aut}G_V$ is a monomorphism.

DEFINITION 4.2.1 (Šmel'kin [24]) The restricted V-verbal wreath product, $A\mathrm{wr}_V B$, of A by B is the splitting extension of G_V by B under θ_V. When V consists of the identity only, $G_V = C$, and the corresponding verbal wreath product is the free wreath product $A\mathrm{wr}_* B$.

The definition includes the (ordinary) wreath product AwrB which corresponds to the special case where $V(C) = \gamma_2(C)$ and G_V becomes the direct product of the A_b.

$M(A\mathrm{wr}_V B)$ is isomorphic to $M(B) \times S(A\mathrm{wr}_V B)(G_V, B)$ by Theorem 4.1.7. It is possible to prove directly from (4.1.11) that $S(A\mathrm{wr}_V B)(G_V, B)$ is isomorphic to $M(A) \times C_V/[C_V, A\mathrm{wr}_* B]$ but the argument is extremely long and complicated. There is a much shorter way. We prove that $A\mathrm{wr}_* B$ is isomorphic to $A * B$ which enables us to find a presentation for $A\mathrm{wr}_V B$ in terms of presentations for A and B rather than for G_V and B. The arguments of Lemma 4.1.6, Theorem 4.1.7 and Theorem 4.1.8, with S replaced by another group, can still be applied.

The multiplicator of AwrB has been found by Blackburn [2] using a mixture of homology theory and the "direct approach" on a presentation for AwrB similar to Schur's handling of $M(A \times B)$. We obtain an equivalent result in Corollary 4.2.10.

A stronger version of (4.2.2) is stated in Hall and Hartley [12].

(4.2.2) $A \mathrm{wr}_* B$ is isomorphic to $A * B$.

Proof It is usual to denote $a\alpha_b$ by a_b, $a \in A$, $b \in B$. Every element of $A\mathrm{wr}_*B$ is a product of cb' where $b' \in B$ and c is a product of elements a_b, $a \in A$, $b \in B$. But $a_b = a_1{}^b$ in $A\mathrm{wr}_*B$ so that A_1 and B generate $A\mathrm{wr}_*B$. Let ω be the epimorphism from $A * B$ onto $A\mathrm{wr}_*B$ induced by α_1 and the identity on B.

An inverse for ω can be found using (4.1.2). The normal closure of A^B of A in $A * B$ is generated by the subgroups $A^b = \{a^b | a \in A\}$, $b \in B$. Clearly $\beta_b : a \mapsto a^b$, $a \in A$ is an isomorphism of A and A^b. Let β be the natural epimorphism of C onto A^B induced by the $\alpha_b{}^{-1}\beta_b : A_b \to A^b$. We have

$$\begin{aligned}(a_b(b'\theta_*))\beta &= a_{bb'}\beta \\ &= a^{bb'} \\ &= (a_b\beta)^{b'} \quad \text{in } A^B .\end{aligned}$$

This implies that $(c(b'\theta_*))\beta = (c\beta)^{b'}$ for all $c \in C$. Thus the map $\tau : cb \mapsto (c\beta)b$ is an epimorphism from $A\mathrm{wr}_*B$ onto $A * B$ by (4.1.2). Now $a\omega\tau = a$ and $b\omega\tau = b$ so that $\omega\tau$ is the identity on $A * B$ and ω is an isomorphism. //

LEMMA 4.2.3 Suppose F_1/R_1 and F_2/R_2 are presentations for A and B. Then F/R is a presentation for $A\mathrm{wr}_V B$ where

(i) $F = F_1 * F_2$

(ii) $R = R_2 S_V$

(iii) $S_V = [R_2, F_1]^F R_1^F R_V$

(iv) $R_V = V(H) \cap [(F_1^{f_2})^H]$ where $H = F_1^{F_2}$.

Proof The group G_V is defined to be C/C_V. Since C_V is invariant under $b\theta_*$, $b \in B$, C_V is normal in $A\mathrm{wr}_* B$ and (4.1.3) shows that $A\mathrm{wr}_* B/C_V$ is the splitting extension of G_V by B under the action

$$\begin{aligned}(cC_V)^b &= c^b C_V \\ &= c(b\theta_*)C_V \\ &= (cC_V)(b\theta_V) \quad \text{by construction of } \theta_V ,\end{aligned}$$

which implies that $A\mathrm{wr}_* B/C_V = A\mathrm{wr}_V B$.

Let ψ_V be the canonical homomorphism from $A\mathrm{wr}_* B$ onto $A\mathrm{wr}_V B$, ψ be the natural homomorphism from $F_1 * F_2$ onto $A * B$ and ω the isomorphism from $A * B$ to $A\mathrm{wr}_* B$ constructed in (4.2.2).

$$F_1 * F_2 \xrightarrow{\psi} A * B \xrightarrow{\omega} A\mathrm{wr}_* B \xrightarrow{\psi_V} A\mathrm{wr}_V B .$$

The kernel of ψ is $R_2^F R_1^F = R_2 [R_2, F_1]^F R_1^F$ by Lemmas 2.2.1 and 2.2.2. But,

$$\begin{aligned}R_V\psi\omega &= (V(D) \cap [(A^b)^D])\omega \quad \text{by Lemma 3.1.4 (ii) where } D = A^B \\ &= C_V \quad \text{by Lemma 3.1.4 (i)} \\ &= \ker\psi_V .\end{aligned}$$

Thus $R = R_2[R_2, F_1]^F R_1^F R_V$ is the kernel of $\psi\omega\psi_V$. //

Compare the statements of (4.1.4) and Lemma 4.2.3. Lemma 4.1.6 and Theorem 4.1.7 are based on (4.1.4). The proofs do not depend on the actual generators of S but only on the facts that S^F is contained in $F_1[F_1,F_2]$ and that $R_1{}^F$ and $[R_2,F_1]^F$ are contained in S^F. S_V also has these properties. We may therefore restate Theorem 4.1.7 in the present context as follows,

(4.2.4) $M(A\mathrm{wr}_V B)$ is isomorphic to $M(B) \times S_V \cap F'/[R_2,F_1][S_V,F]$.

Similarly putting $T_V = S_V\delta$, where δ is the natural homomorphism from F onto $K = F_1 * B$ enables us to restate Theorem 4.1.8 as,

(4.2.5) $S_V \cap F'/[R_2,F_1][S_V,F]$ is isomorphic to $T_V \cap K'/[T_V,K]$.

$T_V \cap K'/[T_V,K]$ can be expressed as the required direct product with the aid of a lemma on free products with amalgamation.

DEFINITION 4.2.6 (i) G is a product of the groups A_λ, $\lambda \in I$, amalgamating D if the A_λ generate G and $A_\lambda \cap A_\mu = D$, $\lambda \neq \mu$.

(ii) G is a <u>free</u> product of the A_λ, $\lambda \in I$, amalgamating D if

(a) G satisfies condition (i)

(b) When $\phi_\lambda : A_\lambda \to \bar{A}$ is an isomorphism and $\bar{G}$ is a product of the $\bar{A}_\lambda$ amalgamating $\bar{D}$ in such a way that $D\phi_\lambda = \bar{D}$ and $d\phi_\lambda = d\phi_\mu$, $d \in D$; $\lambda,\mu \in I$; then there exists a homomorphism of G onto $\bar{G}$ extending the ϕ_λ.

(iii) G is a <u>central</u> product of the A_λ, $\lambda \in I$, amalgamating D if G satisfies (i) and $[A_\lambda,A_\mu] = E$, $\lambda,\mu \in I$; $\lambda \neq \mu$.

Suppose we are given the groups A_λ, $\lambda \in I$, each with a subgroup D_λ such that there exists a group D with isomorphisms $\delta_\lambda : D_\lambda \to D$. Neumann [20] page 527, proves, by identifying d_λ with $d_\lambda\delta_\lambda$, that it is always possible to construct a free product of the A_λ amalgamating D. It is straightforward to show that this group is isomorphic to the quotient of the (absolutely) free product of the A_λ by the normal closure of the elements $(d_\lambda\delta_\lambda\delta_\mu^{-1})d_\lambda^{-1}$, $d_\lambda \in D_\lambda$; $\lambda,\mu \in I$. By the same token, the central product of the A_λ amalgamating D under the δ_λ exists if and only if each D_λ is central in A_λ, [20] page 534.

Lemma 4.2.7 Suppose G is the free product of the A_λ, $\lambda \in I$ amalgamating the subgroups D_λ under the isomorphisms $\delta_\lambda : D_\lambda \to D$. If D_λ is central in A_λ for each $\lambda \in I$, then A_λ intersects $[A_\mu{}^G]$ trivially.

Proof Let ν be the natural homomorphism from G onto the central product of the A_λ amalgamating D under the δ_λ. Theorem 4.6, [29] shows that $\ker\nu = [A_\lambda{}^G]$. But $\ker\nu \cap A = E$ by construction of ν. //

(4.2.8) (i) $T_V \cap K' = (R_1 \cap F_1{}')[R_1,B]^K K_V$

(ii) $[T_V,K] = [R_1,F_1][R_1,B]^K[K_V,K]$

where $K = F_1 * B$, $K_V = V(J) \cap [(F_1{}^b)^J]$ and $J = F_1{}^B$.

Proof By definition,

$$\begin{aligned} T_V &= S_V\delta \\ &= (R_1{}^K R_V)\delta \quad \text{since } [R_2,F_1]^F \leq \ker\delta \\ &= R_1{}^K K_V \quad \text{by Lemma 3.1.4 (ii)} . \end{aligned}$$

Hence,

$$\begin{aligned} T_V \cap K' &= (R_1[R_1,K]K_V) \cap K' \\ &= (R_1 \cap F_1{}'B'[F_1,B])[R_1,K]K_V \\ &= (R_1 \cap F_1{}')[R_1,F_1][R_1,B]^K K_V && \text{by Lemma 2.2.2} \\ &= (R_1 \cap F_1{}')[R_1,B]^K K_V \end{aligned}$$

and,

$$\begin{aligned} [T_V,K] &= [R_1{}^K,K][K_V,K] && \text{by (1.3.17)} \\ &= [R_1,K][K_V,K] && \text{by (1.3.15)} \\ &= [R_1,F_1][R_1,B]^K[K_V,K] . \end{aligned}$$

The main theorem can now be developed.

THEOREM 4.2.9 $M(A\mathrm{wr}_V B)$ is isomorphic to

$$M(A) \times M(B) \times C_V/[C_V, A\mathrm{wr}_* B] \quad \text{where } C_V = V(C) \cap [A_b{}^C] .$$

Proof The procedure is similar to Theorem 4.1.7. We express the canonical homomorphism $\phi : K \to K/[T_V,K]$ as a product of three homomorphisms in order to prove that $T_V \cap K'/[T_V,K]$ is the direct product of $(R_1 \cap F_1{}')\phi$ and $K_V\phi$ which are then shown to be isomorphic to $M(A)$ and $C_V/[C_V,A\mathrm{wr}_* B]$ respectively. The theorem follows by (4.2.4) and (4.2.5).

(i) Let ξ be the natural homomorphism from K onto $(F_1/[R_1,F_1]) * B$ and ρ, η be the canonical homomorphism, $\rho : K\xi \to K\xi/([R_1,B]^K)\xi$ and $\eta : K\xi\rho \to K\xi\rho/[K_V,K]\xi\rho$. Then

$$\ker\xi\rho\eta = [R_1,F_1]^K[R_1,B]^K[K_V,K]$$
$$= [T_V,K] \qquad \text{by (4.2.8) since,}$$
$$[R_1,F_1]^K = [R_1,F_1]^B$$
$$= [R_1,F_1][[R_1,F_1],B]$$
$$\leq [R_1,F_1][R_1,B]^K .$$

Thus $\ker\phi = \ker\xi\rho\eta$ and, if we identify $K\phi$ with $K\xi\rho\eta$, $\phi = \xi\rho\eta$.

Proposition (4.2.8) implies that $(T_V \cap K')\phi = (R_1 \cap F_1')\phi\, K_V\phi$. Put $X = F_1/[R_1,F_1]$. Now suppose $X\rho$ intersects $[(X^b)^Y]\rho$ trivially where $Y = X^B$. Then $(R_1 \cap F_1')\xi\rho$ and $K_V\xi\rho$ intersect trivially as subgroups of $X\rho$ and $[(X^b)^Y]\rho$ respectively. Furthermore, since $\ker\eta$ is a subgroup of $K_V\xi\rho$, $(R_1 \cap F_1')\phi \cap K_V\phi = E$ and $(T_V \cap K')\phi = (R_1 \cap F_1')\phi \times K_V\phi$.

(ii) We verify (i) by proving that $X\rho \cap [(X^b)^Y]\rho = E$. According to (4.2.2), Y is the free product of its subgroups and X^b, $b \in B$. Now $([R_1,B]^K)\xi$ is a subgroup of Y and $[R_1,B]^K = [R_1,B]^J$ by (1.3.15). By definition, $[R_1,B]$ is generated by the elements $[r,b]$ $r \in R$, $b \in B$, so that $([R_1,B]^K)\xi = \mathrm{sgp}((r\xi)^{-1}(r^b)\xi \mid r \in R_1,\ b \in B)^Y$. The quotient $Y/([R_1,B]^K)\xi$ is therefore the generalised free product of the X^b amalgamating the subgroups $(R_1{}^b)\xi$ under the isomorphisms $r^b\xi \mapsto r\xi$, $r \in R_1$. The required intersection is trivial by Lemma 4.2.7 since $(R_1{}^b)\xi$ is central in $(F_1{}^b)\xi$.

(iii) The intersection of $(R \cap F')\xi\rho$ and $([R_1,B]^K[K_V,K])\xi\rho$ is trivial by (i) and (ii) since $([R_1,B]^K[K_V,K])\xi\rho = [K_V,K]\xi\rho \leq K_V\xi\rho$. Thus $(R \cap F')\xi \cap ([R_1,B]^K[K_V,K])\xi$ is a subgroup of $\ker\rho = ([R_1,B]^K)\xi$. But $([R_1,B]^K)\xi$ is a subgroup of $[F_1\xi, B\xi]$ and this cartesian intersects $F_1\xi$ trivially. It follows that $(R \cap F')\xi \cap ([R_1,B]^K[K_V,K])\xi = E$ so $([R_1,B]^K[K_V,K])\xi$ is the kernel of $\rho\eta$, and we have that $(R \cap F')\phi$ is isomorphic to $(R_1 \cap F_1')\xi = M(A)$.

(iv) Let ξ be the natural homomorphism from $K = F_1 * B$ onto $A * B$. We can use the argument of (4.1.9) to prove that the quotient of $T_V \cap K'/[T_V, K]$ $(= (T_V\ K')\phi)$ and $(R_1 \cap F_1{}')[T_V,K]/[T_V,K]$ is isomorphic to $T_V\xi \cap (A * B)'/[T_V\xi, A * B]$. But $(T_V \cap K')\phi = (R_1 \cap F_1{}')\phi \times K_V\phi$ so that

$$K_V\phi \simeq T_V\xi \cap (A * B)'/[T_V\xi, A * B]$$

$$\simeq T_V\xi\omega \cap (A\mathrm{wr}_* B)'/[T_V\xi\omega, A\mathrm{wr}_* B] \quad \text{since } \omega : A * B \to A\mathrm{wr}_* B \text{ is an isomorphism by (4.2.2).}$$

Finally,

$$T_V\xi\omega = (R_1{}^K(V(J) \cap [(F_1{}^b)^J]))\xi\omega \quad \text{where } J = F_1{}^{F_2}, \text{ by (4.2.8)}$$

$$= (V(D) \cap [(A^b)^D])\omega \quad \text{by Lemma 3.1.4 and because } R_1{}^K = \ker\xi$$

$$= C_V \ ,$$

and hence,

$$K_V\phi \simeq C_V/[C_V, A\mathrm{wr}_* B] \quad \text{since } C_V \le (A\mathrm{wr}_* B)' \ . \qquad //$$

Definition 4.2.1 constructs $A\mathrm{wr}_V B$ using the right regular representation of B. This is a particular instance of the situation where B acts as a permutation group on a given set I and $A\mathrm{wr}_V B$ is defined to be the splitting extension of $G_V = \prod^V_{\lambda\in I} A_\lambda$ $(A_\lambda \simeq A)$ by B under the action of B on G_V induced by the isomorphisms $A_\lambda \to A_{\lambda b}$. The above argument carries through, replacing the A_b by the A_λ.

When $G = C/H$ is a regular product of the A_b and $H(B\theta_*) = H$, a splitting extension of G by B can be constructed to form a group similar to $A\mathrm{wr}_V B$. Again the work of this section would need little modification to show the multiplicator of this splitting extension to be $M(A) \times M(B) \times H/[H, A\mathrm{wr}_* B]$.

It may also be possible to treat the case of an unrestricted wreath product [21] 2.2.

There is an interesting alternative proof to Theorem 4.2.9 when A is abelian. The cartesian [A,B] in A * B is generated by the elements $a^{-1}a^{b}$ so that

$$\begin{aligned}[A,B]\omega^{-1} &= \mathrm{sgp}(a_1^{-1}a_b \mid a \in A,\ b \in B) \\ &= \mathrm{sgp}(a_b^{-1}a_{b'} \mid a \in A;\ b,b' \in B) \quad \text{in } A\mathrm{wr}_*B\ .\end{aligned}$$

Now $[A_b{}^C]$ is the normal closure of the $[a_b, a'_{b'}]$ but

$$\begin{aligned}[a_b, a'_{b'}] &= a_b^{-1}a_{b'}'^{-1}a_b a_{b'} \\ &= (a_b^{-1}a_{b'})a_{b'}^{-1}a_{b'}'^{-1}a_b a_b'(a_b'^{-1}a_{b'}')\end{aligned}$$

and

$$\begin{aligned}a_{b'}^{-1}a_{b'}'^{-1}a_b a_b' &= [a_{b'}, a'_{b'}]a_{b'}'^{-1}a_{b'}^{-1}a_b a_b' \\ &= [a,a']_{b'}(aa')_{b'}^{-1}(aa')_b \\ &= (aa')_{b'}^{-1}(aa')_b\end{aligned}$$

for each $a,a' \in A$ if and only if A is abelian,

so that, when A is abelian, $[A_b{}^C]$ is a subgroup of $[A,B]\omega^{-1}$ or equivalently $[A_b{}^C]\omega$ is a subgroup of [A,B]. The first part of the proof of (4.2.2) shows that $A\mathrm{wr}_V B = A * B/C_V\omega$ and C_V is a subgroup of $[A_b{}^C]$. $A\mathrm{wr}_V B$ is therefore a regular product of A and B. In this case Theorem 4.2.9 comes directly from Theorem 2.2.6.

Returning to the general case, we have that $A\mathrm{wr}_*B$ is the product CB so that $[C_V, A\mathrm{wr}_*B] = [C_V,C][C_V,B]$. Let α_V be the canonical homomorphism from $A\mathrm{wr}_*B$ onto $A\mathrm{wr}_*B/[C_V,C]$. Then $C_V/[C_V,A\mathrm{wr}_*B]$ is isomorphic to $C_V\alpha_V/[C_V,B]\alpha_V$ which is a regular product of the A_b since $[C_V,C] \le [A_b{}^C]$.

We can be more specific when $V(C) = \gamma_2(C)$ and $\mathrm{Awr}_V B$ becomes the ordinary wreath product AwrB.

COROLLARY 4.2.10 M(AwrB) is isomorphic to
$M(A) \times M(B) \times (\prod^{\times}_{b_1<b_2} A_{b_1} \otimes A_{b_2})/N$ where "<" is any fixed ordering on the underlying set of B and N is the subgroup generated by the elements,

$$\left.\begin{array}{ll} (a_{b_1} \otimes a'_{b_2})^{-1}(a_{b_1b} \otimes a'_{b_2b}) & b_1b < b_2b \\ & \\ (a_{b_1} \otimes a'_{b_2})(a_{b_1b} \otimes a'_{b_2b}) & b_2b < b_1b \end{array}\right\} a,a' \in A;\ b,b_1,b_2 \in B$$

and $b_1 < b_2$.

Proof From above $M(\mathrm{Awr}_V B) = M(A) \times M(B) \times C_V\alpha_V/[C_V,B]\alpha_V$. In this case $C_V = [A_b{}^C]$ so that $C\alpha_V = \prod^{(2)}_{b\in B} A_b$ and hence $C_V\alpha_V = [A_b{}^G]$ where $G = \prod^{(2)}_{b\in B} A_b$. The structure of $[A_b{}^G]$ is well known. We have that,

(i) $[A_b{}^G]$ is central in G

(ii) $[A_{b_1},A_{b_2}] = [A_{b_2},A_{b_1}]$

(iii) $[A_b{}^G] = \mathrm{sgp}([A_{b_1},A_{b_2}] \mid b_1,b_2 \in B;\ b_1 \in b_2)$ by (i) and (ii)

(iv) $[A_{b_1},A_{b_2}] \cap \mathrm{sgp}([A_{b_3},A_{b_4}] \mid b_3,b_4 \in B;\ b_3 \in b_4;$
$(b_3,b_4) \neq (b_1,b_2)) = E$ by the associativity of nilpotent multiplication.

(v) $[A_{b_1},A_{b_2}] \simeq A_{b_1} \otimes A_{b_2}$ since $\mathrm{sgp}(A_{b_1},A_{b_2}) = A_{b_1}(2)A_{b_2}$.

Therefore $[A_b{}^G] = \prod^{\times}_{b_1<b_2} A_{b_1} \otimes A_{b_2}$.

By (4.1.3) $C_V\alpha_V$ is a splitting extension of G by B under the action $(a_b)^{b'} = a_{bb'}$. Hence

$$[C_V,B]\alpha_V = \mathrm{sgp}([[a_{b_1},a'_{b_2}],b] \mid b_1,b_2,b \in B;\ a,a' \in A;\ b_1 < b_2)$$

$$= \mathrm{sgp}([a_{b_1},a'_{b_2}]^{-1}[a_{b_1b},a'_{b_2b}] \mid b_1,b_2,b \in B;\ a,a' \in A,\ b_1 < b_2).$$

The result follows since $b_1b \neq b_2b$ and $[a_{b_1b},a_{b_2b}] = [a_{b_2b},a_{b_1b}]^{-1}$. //

3. Construction of a representing group for Awr_VB

Given representing pairs (L,M) and (K,N) for A and B it is fairly easy to guess what a representing pair for Awr_VB might look like. Let $\mu_b : L_b \to L$ be an isomorphism and let D be the free product of the L_b amalgamating M under the isomorphisms $\mu_b : M_b \to M$ where $M_b = M\mu_b^{-1}$. By Definition 4.2.6 (ii) (b) the $\mu_b\mu_{bb'}^{-1}$ induce an automorphism $b\theta$ of D. Suppose that $\delta : K \to B$ is the epimorphism with kernel N and that U is the splitting extension of D by K under the action $\delta\theta : K \to \mathrm{Aut}D$. Now $D_V = V(D) \cap [L_b^D]$ is left invariant by the $x\delta\theta$, $x \in K$, so that $[D_V,U]$ is a subgroup of D_V which is also invariant. Note that $M \cap D_V = E$ by Lemma 4.2.6. Proposition (4.1.3) implies that $J = U/[D_V,U]$ is the splitting extension of $G = D/[D_V,U]$ by K under the action $\theta_V : K \to \mathrm{Aut}G$ where $x\theta_V$, restricted to L_b, is $\mu_b\mu_{b(x\delta)}^{-1} : L_b \to L_{b(x\delta)}$. (G is generated by the L_b since $L_b \cap [D_V,U] \leq L_b \cap [L_b^D] = E$ by Lemma 4.2.6).

THEOREM 4.3.1 Suppose Awr_VB is finite and that (L,M) and (K,N) are representing pairs for A and B. Then (J,Q) is a representing pair for Awr_VB where J is as defined above, $Q = MNG_V$ and $G_V = V(G) \cap [L_b^G]$.

Proof Q is a subgroup of $Z(K) \cap K'$ by construction. The subgroup G_V of Q equals $D_V/[D_V,U]$ by Lemma 3.1.4 and $M \cap D_V = E$ in D by Lemma 4.2.6 so that $M \cap G_V = E$ in G. Thus, since J is a splitting extension of G by K,

$$MN \cap G_V = M \cap G_V = E ,$$

$$M \cap NG_V = M \cap G_V = E ,$$

and
$$N \cap MG_V \leq K \cap G = E .$$

This implies that $Q = M \times N \times G_V$ and the result will follow by Theorem 4.2.9 provided we can show that $K/Q \simeq Awr_V B$ and $G_V \simeq C_V/[C_V, Awr_* B]$.

By construction, $J/Q \simeq \dfrac{U/[D_V, D]}{MND_V/[D_V, D]} \simeq U/MND_V$. Now $D/M = C = \prod^*_{b \in B} A_b$ where $A_b = L_b/M_b$ and (4.1.3) implies that U/M is the splitting extension of C by K under $\delta\theta_*$ (see section 2). Proposition (4.1.2) may be applied to show that $\dfrac{U/M}{N}$ is the splitting extension of C by K/N under the action $a_b^{xN} = a_{b(x\delta)}$, $a \in A$, $x \in K$. That is U/MN is isomorphic to $Awr_* B$. Therefore there exists an epimorphism τ from U onto $Awr_* B$ with kernel MN. But $D_V\tau = C_V$ by Lemma 3.1.4 (ii) so that U/MND_V is isomorphic to $Awr_* B/C_V = Awr_V B$.

Finally,

$$\begin{aligned} D_V \cap \ker\tau &= D_V \cap MN \\ &= D_V \cap M \\ &= E \end{aligned}$$

and hence,

$$\begin{aligned} G_V &= D_V/[D_V, U] \\ &\simeq D_V \;/[D_V, U] \\ &= C_V/[C_V, Awr_* B] \quad \text{by Lemma 3.1.4 .} \end{aligned}$$ //

CHAPTER 5

THE SPLITTING EXTENSIONS OF A FINITE ABELIAN GROUP

1. The splitting extensions of one cycle by another

Suppose that A and B are finite cycles generated by elements x and y of orders m and n respectively and that G is the splitting extension of A by B under θ. The homomorphism $\theta : B \to \mathrm{Aut}A$ is induced by the action of y on x in G. This action must be of the form $x(y\theta) = x^{\ell}$ where the highest common factor, (ℓ,m), of ℓ and m is one and m divides $\ell^n - 1$. The multiplicator of G turns out to be the cycle of order $(\ell-1, m, \sum_{i=1}^{n} \ell^{i-1})$.

Firstly $M(G) = U^C \cap C'/[U,C]$ where $C = A * B$ and $U = \mathrm{sgp}(a^{-1}(a(b\theta))[b,a] \mid a \in A,\ b \in B)$ by Corollary 4.1.10 since $M(A) = E$. Let $D = \mathrm{sgp}(x^{\ell-1}[y,x])$. Now, since C is a splitting extension of $A[A,B]$ by B, (4.1.3) may be applied to show that both C/U^C and C/D^C are isomorphic to G. But $D \leq U$ so that $D^C = U^C$ and hence $M(G) = D^C \cap C'/[D,C]$.

For convenience we take C as being generated by x^r and y^s, $1 \leq r \leq m$, $1 \leq s \leq n$. Then, from (1.3.16),

$$\begin{aligned}[D,C] &= \mathrm{sgp}([x^{\ell-1}[y,x],x^r],[x^{\ell-1}[y,x],y^s] \mid 1 \leq r \leq m,\ 1 \leq s \leq n)^C \\ &= \mathrm{sgp}([y,x,x^r],[x^{\ell-1},y^s]^{[y,x]}[y,x,y^s] \mid 1 \leq r \leq m,\ 1 \leq s \leq n)^C\end{aligned}$$

by (1.3.4) .

If ϕ is the canonical homomorphism from C onto $C/[D,C]$ then it follows by induction on r and s that,

$$[y,x^r]\phi = ([y,x]\phi)^r \qquad - (1)$$

$$[y^s,x]\phi = ([y,x]\phi)^{\sigma(s)} \quad \text{where } \sigma(s) = \sum_{i=1}^{s} \ell^{i-1} \qquad - (2)$$

since (1) is trivial for $r = 1$, (2) is trivial for $s = 1$ and,

$$\begin{aligned} [y,x^{r+1}]\phi &= ([y,x^r][y,x][y,x,x^r])\phi \\ &= [y,x^r]\phi[y,x]\phi \qquad - (1)' \end{aligned}$$

$$\begin{aligned} [y^{s+1},x]\phi &= ([y,x][y,x,y^s][y^s,x])\phi \\ &= ([y^s,x^{\ell-1}][y,x][y^s,x])\phi \\ &= ([y^s,x]^{\ell-1}[y,x][y^s,x])\phi . \qquad - (2)' \end{aligned}$$

From above,

$$\begin{aligned} M(G) &= (D^C \cap C')\phi \\ &= (D^C)\phi \cap C'\phi \quad \text{since } \ker\phi \leq [A,B] = C' \\ &= D\phi \cap C'\phi . \end{aligned}$$

Note that $C\phi$ is a regular product of $A\phi$ and $B\phi$. If $z \in D\phi$ then $z = ((x^{\ell-1}[y,x])\phi)^r$, for some r, and hence $z = (x\phi)^{r(\ell-1)}([y,x]\phi)^r$. It follows that $z \in C'\phi$ if and only if $(x\ )^{r(\ell-1)} = 1$. That is $D\phi \cap C'\phi = \mathrm{sgp}([y,x]^t)\phi$ where $t = m/(\ell-1,m)$ (the order of $x^{\ell-1}$).

To find the order of $[A,B]\phi$, ($= \mathrm{sgp}([x,y]\phi)$ by (1) and (2)) we observe that $([y,x]\phi)^m = 1$ and $([y,x]\phi)^{\sigma(n)} = 1$ by (1) and (2). Hence $|[A,B]\phi|$ divides $(m,\sigma(n))$. Conversely $C\phi$ can be reconstructed from H where H is the cycle of order $(m,\sigma(n))$ generated by z. The map $x \mapsto xz$, $z \mapsto z^\ell$, induces an automorphism, α, of $H \times A$ since $z^m = 1$ and $(\ell,\sigma(n)) = 1$. But $x\alpha^r = xz^{\sigma(r)}$ and $z\alpha^r = z^{\ell^r}$ by induction on r. Thus $\alpha^n = 1$ since $(\sigma(n),m)$ divides $\sigma(n)$ and m divides ℓ^n-1. Let K be the splitting extension of $H \times A$ by B under the action induced by the map

$y \mapsto \alpha$. K is generated by A and B so that there exists a natural homomorphism ψ from C onto K. It is easy to check that $[A,B]\phi = H$ and $(\ker\phi)\psi = E$. Thus $(m,\sigma(n))$ divides $|[A,B]\phi|$.

We have that M(G) is a cyclic group of order $u/(u,t)$ where $u = (m,\sigma(n))$. Now $m = (\ell-1,m)t$ and $((\ell-1,m),t) = 1$. Therefore $u = ((\ell-1,m),\sigma(n))(t,\sigma(n))$, (u,t) must be $(t,\sigma(n))$ and M(G) is cyclic of order $(\ell-1,m,\sigma(n))$.

Tahara also proves this result [28] Proposition 9.

2. A conjecture

When A is abelian and finitely generated with no elements of even order then it can be shown that SG(A,B) is a central product of a homomorphic image, $L'\phi$, of M(A) and a certain other group. This is a highly technical result, almost useless for actual computation. However $L'\phi$ is a direct factor of SG(A,B) when G is also a p-group, p odd, (Evens [4]). The conjecture is that $L'\phi$ remains a direct factor when this extra condition is dropped.

We will only sketch the argument. Suppose $A = \prod_{i=1}^{n}{}^{\times} A_i$ where $A_i = \mathrm{sgp}(a_i)$. A slight modification of (4.1.11) yields that SG(A,B) is isomorphic to $W^J \cap J'/[W,J]$ where

(i) $J = L * B$

(ii) $W = \mathrm{sgp}(\ell_1^{-1}\ell_2[b,\ell_1] \mid \ell_1,\ell_2 \in L;\ b \in B;\ \ell_2\alpha = \ell_1\alpha(b\theta))$

(iii) $L = \prod_{i=1}^{n}{}^{(2)} A_i$

(iv) α is the homomorphism from L to A with kernel L'.

Note that L' is isomorphic to $\prod_{i>j}^{\times} A_i \otimes A_j = M(A)$. The hypothesis that no element of A is even order implies that $\prod_{i=1}^{n} a_i^{n_i}$ has the same order when evaluated in A or L and this enables the construction of automorphisms $b\chi$, $b \in B$ of L such that, $b\chi\alpha = b\theta$. (The map $\chi : B \to \mathrm{Aut}L$ is not usually a homomorphism.) These can be used to express W^J as $W^J = [L',B]^J \ \mathrm{sgp}([b,\ell]\xi[b,\ell]^{-1} \mid \ell \in L,\ b \in B)^J$ where $\xi : [L,B] \to L$ is the homomorphism induced by the map $[\ell,b] \mapsto \ell^{-1}(\ell(b\chi))$ of free generators.

It follows that,

(5.2.1) $W^J \cap J'/[W,J] = (L'\ker\eta)[W,J]/[W,J]$ where $\eta = \xi\alpha$.

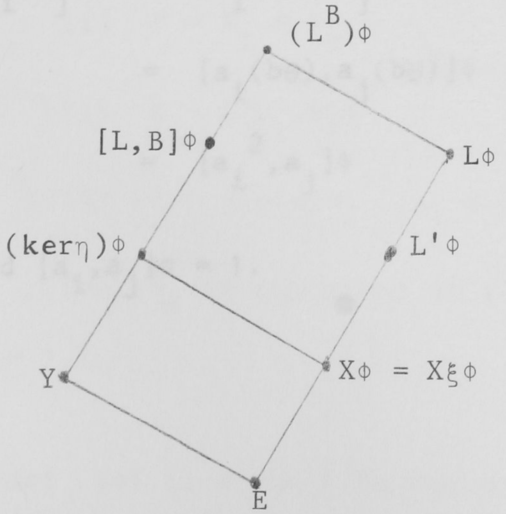
(5.2.2) There is a homomorphism ϕ of J such that $L^B/[W,J] \simeq (L^B)\phi/K\phi$ where

$$K = \mathrm{sgp}([b_1b_2,\ell]^{-1}[b_1,\ell][b_2,\ell(b_1\chi)] \mid b_1,b_2 \in B;\ \ell \in L)^{[L,B]}$$

and $K\phi$ is normal in $J\phi$.

(5.2.3) Suppose $X = [L,B]'\ \mathrm{sgp}([b,\ell_1,\ell_2] \mid b \in B;\ \ell_1,\ell_2 \in L)$. Then the map $\tau : x\phi \mapsto x\xi\phi$, $x \in X$, is an isomorphism and $(L^B)\phi$ is the central product of $[L,B]\phi$ and $L\phi$ amalgamating $X\phi$ under τ.

SG(A,B) is therefore the quotient of a central product of $L'\phi$ and $(\ker\eta)\phi$ by $K\phi$ (K is in $\ker\eta$). In fact $(\ker\eta)\phi = Y \times X\phi$ and



$K\phi \cap X\phi = E$. We do not know whether $K\phi$ is contained in Y so that, in general, SG(A,B) is a central product of $(\ker\eta)\phi/K\phi$ and $L'\phi$ amalgamating $X\phi$ under τ.

Evens states that

$$SG(A,B) = H_1(B,A) \times H_2(A,Z)_B$$

provided that A is abelian and G is a p-group, p odd. But $L'\phi$ is isomorphic to $H_2(A,Z)_B$ in the present context because the natural action of b on $M(A)$ ($= H_2(A,Z)$) corresponds to the restriction of $b\chi$ to L' and $L'\phi$ is actually isomorphic to $L'/[L',B] = L'/\mathrm{sgp}(\ell^{-1}(\ell(b\chi)) \mid \ell \in L', b \in B)$. Therefore, when G is an odd p-group, $L'\phi$ is a direct factor of $SG(A,B)$. This forces $K\phi$ to lie in Y. The fact that G is a p-group does not seem to have any special bearing on the relations involved and this suggests that $K\phi$ may always be contained in Y. If this were the case then the complementary factor of $L'\phi$ in $SG(A,B)$ (i.e. $SG(A,B)/L'\phi$) would be the kernel of the map

$$\left.\begin{array}{l} \zeta : H/N \to A \\ (a,b)N \mapsto a^{-1}(a(b\theta)) \end{array}\right\} \text{ where } H \text{ is the free abelian group}$$

on the pairs (a,b) and N is the subgroup generated by $(aa',b)^{-1}(a,b)(a',b)$, $(a,bb')^{-1}(a,b)(a(b\theta),b')$ $a,a' \in A$; $b,b' \in B$.

The multiplicator of G is $M(B) \times \ker\zeta$ when either A is cyclic (for arbitrary B) or $\theta : B \to \mathrm{Aut}A$ is an endomorphism because $L'\phi$ is trivial in both cases. The first case is obvious. For the second we see that, given i,j there exists a $b\theta$ such that $a_i(b\theta) = a_i^2$ and $a_j(b\theta) = a_j$, $j \neq i$. Therefore,

$$\begin{aligned} [a_i,a_j]\phi &= [a_i(b\chi),a_j(b\chi)]\phi \\ &= [a_i(b\theta),a_j(b\theta)]\phi \quad \text{since } (b\chi)\alpha = b\theta \text{ and } L'\phi \text{ is central} \\ &= [a_i^2,a_j]\phi \end{aligned}$$

and $[a_i,a_j]\phi = 1$. //

CHAPTER 6

CENTRAL PRODUCTS

Broadly speaking we have considered two types of product. When G is generated by A_1 and A_2, and $A_1^G \cap A_2$ and $A_1 \cap A_2^G$ are trivial then $M(A_1) \times M(A_2)$ is a direct factor of $M(G)$. If $A_1 \cap A_2^G$ is non trivial and $A_1^G \cap A_2$ is trivial then at least $M(A_2)$ is a direct factor of $M(G)$. Perhaps the simplest example of a product where neither $A_1^G \cap A_2$ nor $A_1 \cap A_2^G$ is trivial occurs when G is a non degenerate central product of A_1 and A_2 (see Definition 4.2.6 (iii)). It is not surprising that, in this instance, there seems to be no natural embedding of $M(A_1)$ or $M(A_2)$ in $M(G)$. Indeed the usual manoeuvre of constructing a presentation for G in terms of presentations for A_1 and A_2 appears to be inadequate.

Let G be the central product of A_1 and A_2 amalgamating C_1 under the isomorphism $\delta : C_1 \to C_2$ where C_1 and C_2 are central in A_1 and A_2.

(6.1.1) Suppose F_i is a free group and that ν_i is an epimorphism from F_i onto A_i with kernel R_i, $i = 1,2$. Then F/R is a presentation for G where

(i) $F = F_1 * F_2$

(ii) $R = T[F_1,F_2]$

(iii) $T = sgp(s_1 s_2^{-1} | s_1 \in S_1,\ s_2 \in S_2,\ s_1\nu_1\delta = s_2\nu_2)$

(iv) S_i is the group of elements in F_i which maps onto C_i under ν_i, $i = 1,2$.

Proof Let ξ, ρ and η be the natural epimorphisms,

$$F \xrightarrow{\xi} F_1 \times F_2 \xrightarrow{\rho} A_1 \times A_2 \xrightarrow{\mu} G .$$

R_1 and R_2 are obviously contained in T. It is easily checked that $\ker\xi = [F_1,F_2]$, $\ker\rho = R_1R_2$ and $\ker\eta = T\xi\rho$. Therefore $\ker\xi\rho\eta = T[F_1,F_2]$. //

(6.1.2) The multiplicator of G is isomorphic to $H[F_1,F_2]/K$ where H,K and $[F_1,F_2]$ are subgroups of $F_1(2)F_2$ and

(i) $H = \mathrm{sgp}(s_1s_2^{-1} \mid s_1 \in S_1 \cap F_1', s_2 \in S_2 \cap F_2', s_1\nu_1\delta = s_2\nu_2)$

(ii) $K = \mathrm{sgp}([s_1,f_2][s_2,f_2]^{-1}, [s_2,f_1][s_1,f_1]^{-1} \mid s_i \in S_i, f_i \in F_i, i = 1,2, s_1\nu_1\delta = s_2\nu_2)$.

Proof The fact that $F/[F_1,F_2]$ is isomorphic to $F_1 \times F_2$ implies that each element of $T[F_1,F_2]$ is of the form $s_1s_2^{-1}u$ where $s_1 \in S_1$, $s_2 \in S_2$, $u \in [F_1,F_2]$, $s_1\nu_1\delta = s_2\nu_2$. An application of (2.1.3) yields $T[F_1,F_2] = H_0[F_1,F_2]$ where $H_0 = \mathrm{sgp}(s_1s_2^{-1} \mid s_1 \in S_1 \cap F_1', s_2 \in S_2 \cap F_2', s_1\nu_1\delta = s_2\nu_2)$ in F. Let ϕ be the natural homomorphism from F onto $F_1(2)F_2$, then $[T[F_1,F_2],F] = [T,F][[F_1,F_2],F]$ and $\ker\phi = [[F_1,F_2],F]$. Thus $M(G)$ is isomorphic to $(H_0[F_1,F_2])\phi/[T,F]\phi$.

Now $[T,F]$ is generated by the elements $[s_1s_2^{-1},f_1]^x, [s_1s_2^{-1},f_2]^x$ where $x \in F$, $s_i \in S_i$, $f_i \in F_i$, $i = 1,2$, $s_1\nu_1\delta = s_2\nu_2$, by (1.3.16). The commutator $[s_1s_2^{-1},f_1]^x$ can be expanded as $[s_1,f_1]^y[s_2,f_1]^{-y}$ where $y = s_2^{-1}x$ by (1.3.4) and (1.3.3). But both $[s_1,f_1,y]$ and $[s_2,f_1]$ are in $[F_1,F_2]\phi$ which is central in $F_1(2)F_2$ so that $[s_1s_2^{-1},f_1]\phi = ([s_1,f_1][s_2,f_1]^{-1})\phi$. Similarly $[s_1s_2^{-1},f_2]\phi = ([s_1,f_2][s_2,f_1]^{-1})\phi$. //

The subgroup of $F_1(2)F_2$ generated by $S_1 \cap F_1'$ and $S_2 \cap F_2'$ is also their direct product. This means that H is a sort of "diagonal" subgroup of $(S_1 \cap F_1') \times (S_2 \cap F_2')$. K is the direct product of $K_1 \times K_2$ where $K_j = \mathrm{sgp}([s_1,f_j][s_2,f_j]^{-1} \mid s_1 \in S_1, s_2 \in S_2, f_j \in F_j, s_1\nu_1\delta = s_2\nu_2)$, $j = 1,2$, so that K_1 and K_2 are diagonal subgroups of $(S_1 \cap F_1') \times [F_1,F_2]$ and $(S_2 \cap F_2') \times [F_1,F_2]$ respectively.

To get some idea of what can happen when K is factored out of $H[F_1,F_2]$, suppose that C_i is contained in A_i', $i = 1,2$. Then $S_i = (S_i \cap F_i')R_i$ and $[S_i,F_j] = [R_i,F_j]$, $i \neq j$, since $[F_i',F_j]$ is trivial in $F_1(2)F_2$. But $[R_i,F_j]$ is a subgroup of K. Thus $K = [S_1,F_1] \times [S_2,F_2] \times ([R_1,F_2][R_2,F_1])$ and, using the fact that $[F_1,F_2]/[R_1,F_2][R_2,F_1]$ is isomorphic to $A_1 \otimes A_2$, we have,

$$H/K \simeq (H/([S_1,F_1] \times [S_2,F_2])) \times (A_1 \otimes A_2) .$$

Moreover the map $\alpha_i : s_i[S_i,F_i] \mapsto s_i\nu_i$ is an epimorphism from $M(A_i/C_i)$ onto C_i since F_i/S_i is a presentation for A_i/C_i and,

$$\begin{aligned} C_i &\simeq S_i/R_i \\ &= (S_i \cap F_i')R_i/R_i \\ &\simeq (S_i \cap F_i')/(S_i \cap F_i') \cap R_i \\ &= S_i \cap F_i'/R_i \cap F_i' \\ &\simeq \frac{S_i \cap F_i'/[S_i,F_i]}{R_i \cap F_i'/[S_i,F_i]} \qquad [S_i,F_i] \leq R_i \text{ because } C_i \leq Z(A_i). \end{aligned}$$

The kernel of α_i is $R_i \cap F_i'/[S_i,F_i]$ - a homomorphic image of $M(A_i)$. Let β be the homomorphism,

$$\left.\begin{aligned} M(A_1/C_1) \times M(A_2/C_2) &\to C_2 \\ x_1x_2 &\mapsto (x_1\alpha_1\delta)(x_2\alpha_2) \end{aligned}\right\}$$

Then $\ker\beta = H/([S_1,F_1] \times [S_2,F_2])$ and M(G) is isomorphic to $\ker\beta \times (A_1 \otimes A_2)$.

It may be possible to find more information about M(G) by constructing a different presentation for G. One alternative is to take F_i as the group freely generated by a set of elements $f_i(a)$, $a \in A_i$, in one-to-one correspondence with A_i. The map $f_1(c_1) \mapsto f_2(c_1\delta)$ induces an isomorphism between $H_1 = \mathrm{sgp}(f_1(c_1) \mid c_1 \in C_1)$ and $H_2 = \mathrm{sgp}(f_2(c_2) \mid c_2 \in C_2)$. The free product F of F_1 and F_2 amalgamating H_1 under this isomorphism is a free group such that $F/R_1R_2[F_1,F_2]$ is isomorphic to G. Since the amalgamation is already built in, diagonal subgroups like H do not arise. On the other hand $R_1R_2[F_1,F_2] \cap F'$ is difficult to calculate because (2.1.3) is no longer applicable. Perhaps a suitable choice of coset representatives of C_i and F_i/F_i' in F_i will yield some answers. //

REFERENCES

[1] Michael Barr and George S. Reinhart, Cohomology as a derived functor of derivations, Trans. Amer. Math. Soc. 122 (1966), 416-426.

[2] Norman Blackburn, Some homology groups of wreathe products, Illinois J. Math. 16 (1972), 116-129.

[3] N. Burgoyne and P. Fong, The Schur multipliers of the Mathieu groups, Nagoya J. Math. 27 (1966), 733-745.

[4] Leonard Evens, The Schur multiplier of a semi-direct product, Illinois J. Math. 16 (1972), 166-181.

[5] W. Gaschütz, J. Neubüser and Ti Yen, Über den Multiplikator von p-Gruppen, Math. Z. 100 (1967), 93-96.

[6] O.N. Golovin, Nilpotent products of groups, Amer. Math. Soc. Transl. (2) 2 (1956), 89-115.

[7] J.A. Green, On the number of automorphisms of a finite group, Proc. Roy. Soc. A 237 (1956), 574-581.

[8] Karl W. Gruenberg, Cohomological topics in group theory, (Lecture Notes in Mathematics, 143. Springer-Verlag, Berlin, Heidelberg, New York, 1970).

[9] Marshall Hall, Jr., A basis for free Lie rings and higher commutators in free groups, Proc. Amer. Math. Soc. 1 (1950), 575-581.

[10] Marshall Hall, Jr., The Theory of Groups (The Macmillan Company, New York, 1959).

[11] P. Hall, A contribution to the theory of groups of prime power order, Proc. London Math. Soc. 36 (1934), 29-95.

[12] P. Hall and B. Hartley, The stability group of a series of subgroups, ibid (3) 16 (1966), 1-39.

[13] Heinz Kopf, Fundamentalgruppe and zweite Bettische Gruppe, Comment. Math. Helv. 14 (1942), 257-309.

[14] B. Huppert, Endliche Gruppen I (Die Grundlehren der mathematischen Wissenschaften, Band 134. Springer-Verlag, Berlin, Heidelberg, New York, 1967).

[15] Michael R. Jones and James Wiegold, Some results on the multiplicators of finite groups, unpublished.

[16] A.G. Kuroš, The Theory of Groups (OGIZ Moscow-Leningrad 2nd ed., 1953 [Russian]).

[17] Wilhelm Magnus, Abraham Karass and Donald Solitar, Combinatorial Group Theory (Pure and Applied Mathematics 20. Interscience [John Wiley and Sons], New York, London, Sydney, 1966).

[18] S. Moran, Associative operations on groups I, Proc. London Math. Soc. (3) 6 (1956), 581-596.

[19] S. Moran, Associative operations on groups II, ibid (3) 8 (1958), 548-568.

[20] B.H. Neumann, An essay on free products of groups with amalgamations, Philos. Trans. Roy. Soc. London Ser. A 246 (1954), 503-554.

[21] Hanna Neumann, Varieties of Groups (Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 37. Springer-Verlag, Berlin, Heidelberg, New York, 1967).

[22] J. Schur, Über die Darstellung der endliche Gruppen durch gebrochene linear Substitutionen, J. reine angew. Math. 127 (1904), 20-50.

[23] J. Schur, Untersuchungen über die Darstellungen der endlichen Gruppen durch gebrochene linear Substitutionen, ibid 132 (1907), 85-137.

[24] A.L. Šmel'kin, Wreath products and varieties of groups, Soviet Math. Dokl. 5 (1964), 1099-1010.

[25] Robert Steinberg, Algebraic groups and finite groups, Illinois J. Math. 13 (1969), 81-86.

[26] Ruth Rebekka Struik, On nilpotent products of cyclic groups, Can. J. Math. 12 (1960), 447-462.

[27] Ruth Rebekka Struik, On nilpotent products of cyclic groups II, ibid 13 (1961), 557-568.

[28] Ken-Ichi Tahara, On the second cohomology groups of semi-direct products, unpublished.

[29] James Wiegold, Nilpotent products of groups with amalgamations, Publ. Math. Debrecen 6 (1959), 131-168.

[30] James Wiegold, Multiplicators and groups with finite central factor groups, Math. Z. 89 (1965), 345-347.

[31] James Wiegold, Commutator subgroups of finite p-groups, J. Austral. Math. Soc. 10 (1969), 480-484.

[32] James Wiegold, The multiplicator of a direct product, Quart. J. Math. Oxford (2) 22 (1971), 103-105.

[33] James Wiegold, Some groups with non-trivial multiplicators, Math. Z. 120 (1971), 307-308.